# Blake Farman

Research Statement

Mathematics and Statistics
Louisiana Tech University
https://farmanb.github.io


# Background

## Derived Categories and Fourier-Mukai Transformations

Derived categories were initially conceived by Grothendieck as a device for maintaining cohomological data during his reformulation of algebraic geometry through scheme theory, and were fleshed out by his student, Verdier, in his thesis [Ver77]. While originally devised as a mere book keeping device, over time these objects have been recognized as the key to linking algebraic geometry to a broad range of subjects, both within and without mathematics. As such, the study of derived categories has risen to prominence as a central subfield of algebraic geometry.

As is the case for any category associated to a mathematical object, a natural question that arises is this: how much information about the original object is stored in the category? For schemes $X$ and $Y$, this naturally manifests itself as a question asking how much information about the original schemes can be extracted from an equivalence, $\mathrm{D}(X) \to \mathrm{D}(Y)$, between their derived categories of quasi-coherent sheaves. For a general functor, one quickly finds oneself grasping at little more than abstract nonsense. While this may initially appear disheartening, history suggests a plausible attack.

Indeed, one takes inspiration from the simpler case of unital rings and their categories of modules. While it is known in general that non-isomorphic rings may generate equivalent categories of modules, the method of attack by Morita [Mor58] yields a surprising classifying result: any additive equivalence of module categories is naturally isomorphic to tensoring with a bimodule.

With this in mind, one constructs the analogous geometric functor, called a *Fourier-Mukai transform*, with a *kernel* $K \in \mathrm{D}(X \times Y)$ playing the role of the bimodule, and poses the question anew: are all such equivalences of this form? Relative to a generic functor between derived categories, this is a remarkably simple object, and a positive answer would be incredibly powerful. Unfortunately, the structure of the derived category is too pathological to admit such a statement. However, if one is willing to shift perspective by passing to a higher categorical structure, this becomes true in the more abstract context under natural assumptions on the schemes.

## Noncommutative Projective Schemes

Suppose, for example, that one is interested in studying a commutative graded $k$-algebra, $A = k \oplus A_1 \oplus \cdots$, that is finitely generated in degree one. The edicts of modern algebraic geometry suggest that one should consider passing to the projective space $X = \operatorname{Proj} A$, and studying its category $\operatorname{Qcoh} X$ of quasi-coherent sheaves. If one wishes to relax the condition that $A$ is commutative, then unfortunately many of these constructions become inaccessible in general.

However, a famous result of Serre suggests a path: $\operatorname{Qcoh}(X)$ is equivalent to the quotient of the category of graded $A$-modules, $\operatorname{Gr}(A)$, by the subcategory of torsion modules, Tors $(A)$, in the sense of [Gab62]. In the noncommutative situation, these categories retain the same properties as their commutative counterparts, leading Artin and Zhang [AZ94] to define the category of quasi-coherent sheaves on a Noncommutative Projective Scheme to be the category $\mathrm{QGr}\,(A) = \mathrm{Gr}\,(A) \,/\, \mathrm{Tors}\,(A)$ in an effort to harness the power of the modern geometric approach in this noncommutative setting.

While these schemes do not, in general, admit a space on which to do traditional geometry, they have proven effective in providing a way to adapt familiar geometric tools from the commutative setting to the study of noncommutative algebras. If one subscribes to this principle, then their importance in the commutative setting should suggest that derived categories will play a leading role in this study. However, developments in this area are conspicuously absent, suggesting that, as in the commutative setting, the primary stumbling block is the absence of Fourier-Mukai kernels. Having such a statement for the case of noncommutative projective schemes therefore seems of high priority.

# Past Accomplishments

## Fourier-Mukai Kernels for Noncommutative Projective Schemes

In order to attack the problem of providing Fourier-Mukai kernels for noncommutative projective schemes, it is natural to abstract the problem to the higher categorical structure of differential graded (dg) categories. Indeed, working within the homotopy category of the 2-category of all small dg categories over a commutative ring, $\mathrm{Ho}\,(\mathrm{dgcat}_k)$, we gain access to the incredibly elegant reformulation of Fourier-Mukai transformations at the level of pre-triangulated dg categories through the framework of Toën's derived Morita theory [Toë07].

Consider first the case of varieties $X$ and $Y$, for which Toën provides two critical pieces of data:

1. (**existence of an internal Hom**) the localization of the category of all small dg categories at quasi-equivalences, $\mathrm{Ho}\,(\mathrm{dgcat}_k)$, admits an internal Hom, $\mathbf{R}\underline{\mathrm{Hom}}$, and
2. (**geometric recognition**) the subcategory of the Hom between the dg enhancements of $\mathrm{D}(X)$ and $\mathrm{D}(Y)$ consisting of quasi-functors commuting with coproducts is isomorphic in $\mathrm{Ho}\,(\mathrm{dgcat}_k)$ to the enhancement of the derived category of the product, $X \times Y$,
$$\mathbf{R}\underline{\mathrm{Hom}}_c\left(\mathcal{D}(X), \mathcal{D}(Y)\right) \cong \mathcal{D}\left(X \times Y\right).$$

It is first important to observe what data Toën's machinery does and does not provide. For a general triangulated functor, $F\colon \mathrm{D}(X) \to \mathrm{D(Y)}$, (1) yields no new geometric information in that it provides no guidance on establishing (2). However, if the functor in question admits a lift to a dg quasi-functor, then by (2) it must be an integral transform and, moreover, it must be geometric in origin.

Complementing the machinery of Toën, Lunts and Orlov have established that triangulated equivalences between derived categories of abelian categories lift to quasi-equivalences of their associated dg categories [LO10]. For varieties, in light of geometric recognition, the combination of these two results states that triangulated equivalences $F\colon \mathrm{D}(X) \to \mathrm{D}(Y)$ are necessarily geometric in origin.

Within the realm of noncommutative projective schemes, combining Toën's internal Hom with Lunts and Orlov's uniqueness of differential graded enhancements immediately allows one to conclude that any equivalence $F\colon \mathrm{D}(X) \to \mathrm{D}(Y)$ yields a quasi-equivalence $\mathcal{F}\colon \mathcal{D}(X) \to \mathcal{D}(Y)$ at the differential graded level. Unfortunately, as in the case of varieties, one obtains no new information by simply viewing this equivalence as an object of the highly abstract internal Hom category. One therefore requires a noncommutative projective analogue of geometric recognition.

In this direction, the most basic questions with which one must grapple are:

**Question.**

1. For noncommutative projective schemes, $X$ and $Y$, what noncommutative projective scheme plays the role of the product, $X \times Y$?
2. What is an integral transform in noncommutative projective geometry?
3. Does geometric recognition hold for $X$ and $Y$ (and $X \times Y$)?

For the first, there really can be only one honest noncommutative projective scheme deserving of the name: the Segre product. The second remains separate from the dg structure, and no such creature has been observed in the literature. However, even a glance at the simpler question of graded Morita theory [Zha96] indicates that the situation is already more complicated for noncommutative projective schemes.

Finally, the answer to geometric recognition is positive under cohomological restrictions on $X$ and $Y$. We provide these conditions on a pair of rings, which we refer to as a *delightful couple*, a notion of integral transform, and establish geometric recognition for noncommutative projective schemes under these conditions in [BF21]. A version of the main theorem from the article is

**Theorem** ([BF21])**.** *Let $X$ and $Y$ be noncommutative projective schemes associated to a delightful couple, $A$ and $B$, over a field, $k$. If $A$ and $B$ are both generated in degree one, then geometric recognition holds for $X$ and $Y$. That is, there exists a quasi-equivalence*

$$\mathbf{R}\underline{\mathrm{Hom}}_c(\mathcal{D}(X), \mathcal{D}(Y)) \cong \mathcal{D}(X \times Y).$$

This geometric recognition holds for a general delightful couple, although one must step slightly outside the bounds of noncommutative projective schemes, without losing the (noncommutative) geometry, to obtain the correct product.

As an immediate corollary, one obtains that equivalences between noncommutative projective schemes are necessarily (noncommutative) geometric in nature, along the lines of Rickard [Ric89] or Orlov [Orl97]. Note that this statement makes no reference to differential graded categories, and one recovers the analogous result for projective varieties by restricting to commutative rings.

**Theorem** ([BF21])**.** *Let $X$ and $Y$ be noncommutative projective schemes associated to a delightful couple, $A$ and $B$, over a field, $k$. If there is a triangulated equivalence $F\colon \mathrm{D}(X) \to \mathrm{D}(Y)$, then there exists an object $K$ of $\mathrm{D}(X \times Y)$ whose associated integral transform, $\Phi_K$, is an equivalence. That is, $X$ and $Y$ are Fourier-Mukai partners.*

## Future Work

As the introduction of Fourier-Mukai kernels to noncommutative algebraic geometry forms only the foundation for a theory of derived categories in this new setting, there are a great many questions suggested by the commutative case that need to be addressed. Moving forward, the goal is to begin making progress on the following

> **Question.** How far away from isomorphic are noncommutative Fourier-Mukai partners?

In this direction, one should first understand the situation for noncommutative surfaces, which are the simplest properly noncommutative projective schemes. In the commutative case, the situation is relatively straight-forward

**Proposition** ([Huy06, Prop 12.28])**.** A surface $X$ admits only a finite number of Fourier-Mukai partners.

Unfortunately, this result relies on the classification of minimal surfaces, which at present is still an open problem for noncommutative surfaces [Art97], making it appear too optimistic to hope for such a result at present.

Instead, we restrict ourselves to the class of noncommutative surfaces analogous to $\mathbb{P}^2$. These schemes have been completely classified [ATVdB07, Ste96, Ste97] and arise from the Artin-Schelter regular algebras [AS87] of Gelfand-Kirillov dimension 3 with Hilbert series $(1-t)^{-3}$ [SVdB01, Section 11]. These surfaces fall broadly into two categories [Sta02]: those that are an honest $\mathbb{P}^2$ in the sense that $\operatorname{qgr} A$ is equivalent to the coherent sheaves on a commutative $\mathbb{P}^2$, and those whose point modules are parameterized by an elliptic curve, the latter containing the Sklyanin algebras. The primary goal of this project is to prove the following

> **Objective.** If $X$ and $Y$ are derived equivalent noncommutative $\mathbb{P}^2$s, then $X \cong Y$.

As our framework allows us to regard traditional varieties as a special case of noncommutative projective schemes, this objective is motivated by the special case of [Huy06, Prop 12.28] due to Bondal and Orlov.

**Theorem** ([BO95])**.** *Let $X$ and $Y$ be smooth projective varieties and assume that the (anti-)canonical bundle of $X$ is ample. If there exists an exact equivalence $D^b(X) \simeq D^b(Y)$, then $X$ and $Y$ are isomorphic.*

Encouragingly, the main tool for the reconstruction of these varieties is the classification of point objects in the derived category, which appears to align well with the point modules that are used to classify the noncommutative $\mathbb{P}^2s$. In order to tackle the larger reconstruction theorem, there are natural questions that arise along the way to be addressed in several phases of the research program.

The first is to focus on a systematic study of the derived categories of the noncommutative projective schemes associated to non-degenerate Sklyanin algebras. During this phase, work will be focused on answering the question of whether the point modules of a Sklyanin algebra form a spanning class for the derived category. The goal is either to prove the point modules form a spanning class for all Sklyanin algebras or to provide explicit counter-examples and reasonable conditions under which the point modules form a spanning class.

The second phase of the program is to classify the point objects in the derived category of the noncommutative projective schemes associated to non-degenerate Sklyanin algebras. Positive results for this specific class of noncommutative surfaces will shed light on the situation for more generic cases of noncommutative surfaces. The hope is this will lead to a general classification of point objects in the derived category for noncommutative projective schemes.

In the third phase, it is expected that classification results from the second phase will characterize the point objects of the derived category as the point modules associated to the Sklyanin algebras. Furthermore, the classification of point objects and a reconstruction result for these simpler objects will hopefully lead to natural generalizations to broader classes of noncommutative projective schemes.

# Student Involvement

## Graduate Students

For students who are interested in pursuing work in (noncommutative) algebraic geometry, the future work proposed above would provide multiple avenues for fruitful engagement. These problems are multi-faceted and would provide students with opportunities both to grapple with the theoretical apparatus and with computational approaches. As an example, students could utilize software like SageMath to construct novel tools that can be used to search classes of Artin-Schelter regular algebras for non-trivial Fourier-Mukai partners.

For students with an eye towards more applied projects, recent mathematical developments suggest novel approaches to using my work to bridge the gap between mathematics and computer science. Some examples include using enriched categories to study natural language processing [BTV22] and methods of homological algebra and algebraic topology to study data science [Oud15].

## Undergraduate and Master's Students

During my time at the University of Louisiana at Monroe, I have engaged students in short term research projects through the Emerging Scholars program, and have several projects in the pipeline for future students. These projects come from areas of mathematics that are accessible to students with a variety of backgrounds, such as discrete mathematics, formalization of mathematical results using the Lean proof assistant, knot theory, and tropical geometry. These projects provide students with the opportunity to engage with mathematics beyond what is available in the curriculum and to experience first-hand what it is like to be a research mathematician. The scope of these projects can be tailored to suit the goals of the individual student and range from an expository work in which the student aims to expand his or her own knowledge, to a more lengthy senior capstone project, to generating original results that would be suitable for publication in an undergraduate journal or as a master's thesis.

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