EXAM 3

BLAKE FARMAN

Lafayette College

Answer the questions in the spaces provided on the question sheets and turn them in at the end of the exam period.
It is advised, although not required, that you check your answers. All supporting work is required. Unsupported or otherwise mysterious answers will **not receive credit.**
You may **not** use a calculator or any other electronic device, including cell phones, smart watches, etc.

Name: Solutions

Functions

1. Find the value of b that makes the function

$$f(x) = \begin{cases} \dfrac{x^2-16}{x+4} & \text{if } -4 < x \\ x^2 + b & \text{if } x \leq -4 \end{cases}$$

a continuous function.

$$\lim_{x \to -4^-} f(x) = \lim_{x \to -4^-} \frac{(x-4)(x+4)}{x+4} = \lim_{x \to -4^-} x-4 = -4-4 = -8$$

$$\lim_{x \to -4^+} f(x) = \lim_{x \to -4^+} x^2 + b = (-4)^2 + b = 16 + b = f(-4)$$

$$16 + b = -8 \Rightarrow b = -8 - 16 = \boxed{-24}$$

Date: August 2, 2019.

Derivatives

2. Use the **limit definition**

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

to compute the derivative of

$$f(x) = \sqrt{x+2}.$$

$$\lim_{h\to 0} \frac{\sqrt{x+h+2} - \sqrt{x+2}}{h} = \lim_{h\to 0} \frac{\sqrt{x+h+2} - \sqrt{x+2}}{h} \frac{\sqrt{x+h+2} + \sqrt{x+2}}{\sqrt{x+h+2} + \sqrt{x+2}}$$

$$= \lim_{h\to 0} \frac{(x+h+2) - (x+2)}{h\left(\sqrt{x+h+2} + \sqrt{x+2}\right)}$$

$$= \lim_{h\to 0} \frac{h}{h\left(\sqrt{x+h+2} + \sqrt{x+2}\right)}$$

$$= \lim_{h\to 0} \frac{1}{\left(\sqrt{x+h+2} + \sqrt{x+2}\right)}$$

$$= \frac{1}{\sqrt{x+0+2} + \sqrt{x+2}}$$

$$= \boxed{\frac{1}{2\sqrt{x+2}}}$$

Derivative Rules

3. Compute the line tangent to

$$f(x) = x^3 - x$$

at the point $(1, 0)$.

$f'(x) = 3x^2 - 1$

$f'(1) = 3 - 1 = 2$

$y - 0 = 2(x-1)$

$\boxed{y = 2x - 2}$

Product and Quotient Rules

4. Compute
$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\sin(x)\cos(x)\right].$$

$$\frac{d}{dx}\left[\sin(x)\cos(x)\right] = \frac{d}{dx}\left(\sin(x)\right)\cos(x) + \sin(x)\frac{d}{dx}\left(\cos(x)\right)$$
$$= \boxed{\cos^2(x) - \sin^2(x)}$$

5. Compute
$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{3x+2}{\cos(x)}\right].$$

$$\frac{\frac{d}{dx}(3x+2)\cdot\cos(x) - (3x+2)\frac{d}{dx}\cos(x)}{\cos^2(x)} = \frac{3\cos(x) - (3x+2)(-\sin(x))}{\cos^2(x)}$$
$$= \boxed{\frac{3\cos(x) + (3x+2)\sin(x)}{\cos^2(x)}}$$

Chain Rule

6. Compute

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\sec\left(x^2\right)\right].$$

$$\frac{d}{dx}\left[\sec(x^2)\right] = \sec(x^2)\tan(x^2)\frac{d}{dx}(x^2)$$

$$= \boxed{2x\sec(x^2)\tan(x^2)}$$

Implicit Differentiation and Related Rates

7. Gas is escaping a spherical balloon at the rate of 4 cm^3 per minute. How fast is the surface area shrinking when the radius is 24 cm? For a sphere, $V = \frac{4}{3}\pi r^3$ and $S = 4\pi r^2$ where V is volume, S is surface area and r is the radius of the balloon.

$$-4 = \frac{dV}{dt}$$

$$= \frac{4}{3}\pi\left(3r^2 \, dr/dt\right)$$

$$= 4\pi r^2 \frac{dr}{dt}$$

$$\Rightarrow \frac{dr}{dt} = \frac{-4}{4\pi r^2} = \frac{-1}{\pi r^2}$$

$$\frac{dS}{dt} = 4\pi\left(2r \, dr/dt\right)$$

$$= 8\pi r \frac{dr}{dt}$$

$$= 8\pi r\left(\frac{-1}{\pi r^2}\right)$$

$$= \frac{-8}{r} = \frac{-8}{24} = \boxed{\frac{-1}{3} \text{ cm/min}}$$

Derivatives and Shape

Use the function

$$f(x) = x^3 - 4x$$

to answer the following questions.

8. Find the intervals where the function is increasing and decreasing. Write NONE if not applicable.

Increasing: $\left(-\infty, -\frac{2\sqrt{3}}{3}\right) \cup \left(\frac{2\sqrt{3}}{3}, \infty\right)$

Decreasing: $\left(-2\sqrt{3}/3, \frac{2\sqrt{3}}{3}\right)$

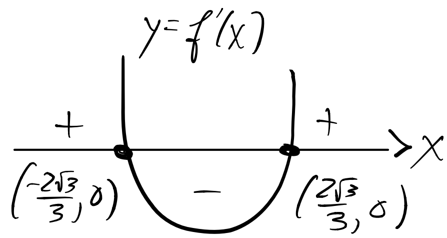
$$f(x) = x(x^2-4) = x(x+2)(x-2)$$

$$f'(x) = 3x^2 - 4 = 0$$

$$\Rightarrow 3x^2 = 4$$

$$\Rightarrow x^2 = 4/3$$

$$\Rightarrow x = \pm\sqrt{4/3} = \pm\frac{2}{\sqrt{3}} = \pm\frac{2\sqrt{3}}{3}$$



9. State the local maximum and local minimum value(s). Write NONE if not applicable.

Local maximum value(s): $x = -\frac{2\sqrt{3}}{3}$

Local minimum value(s): $x = \frac{2\sqrt{3}}{3}$

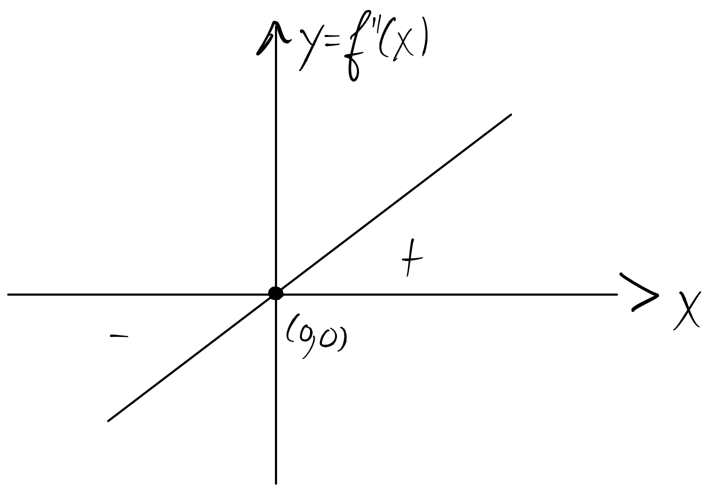
10. Find the intervals on which the function is concave up and concave down. State the inflection points. Write NONE if not applicable.

Concave Up: $(0, \infty)$

Concave Down: $(-\infty, 0)$

Inflection Points: $(0, 0)$

$$f''(x) = 6x = 0 \Rightarrow x = 0$$



Asymptotes

11. Find the asymptotes of

$$f(x) = \frac{2x^2 + 2x - 12}{x^2 - 9}$$

Write NONE if there are none.

Horizontal: $y = 2$

Vertical: $x = 3$

$$\lim_{x \to \infty} \frac{2x^2+2x-12}{x^2-9} = \lim_{x \to \infty} \frac{x^2(2 + 2/x - 12/x^2)}{x^2(1 - 9/x^2)}$$

$$= \lim_{x \to \infty} \frac{2 + \frac{2}{x} - \frac{12}{x^2}}{1 - 9/x^2}$$

$$= \frac{2+0+0}{1-0}$$

$$= 2$$

$$f(x) = \frac{2(x^2+x-6)}{(x+3)(x-3)} = \frac{2(x+3)(x-2)}{(x+3)(x-3)}$$

$$\lim_{x \to -3} \frac{2(x+3)(x-2)}{(\cancel{x+3})(x-3)} = \lim_{x \to -3} \frac{2(x-2)}{(x-3)} = \frac{2(-3-2)}{-3-3} = \frac{-10}{-6} = -\frac{5}{3}$$

$$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} \frac{2(x-2)}{x-3} = \infty$$

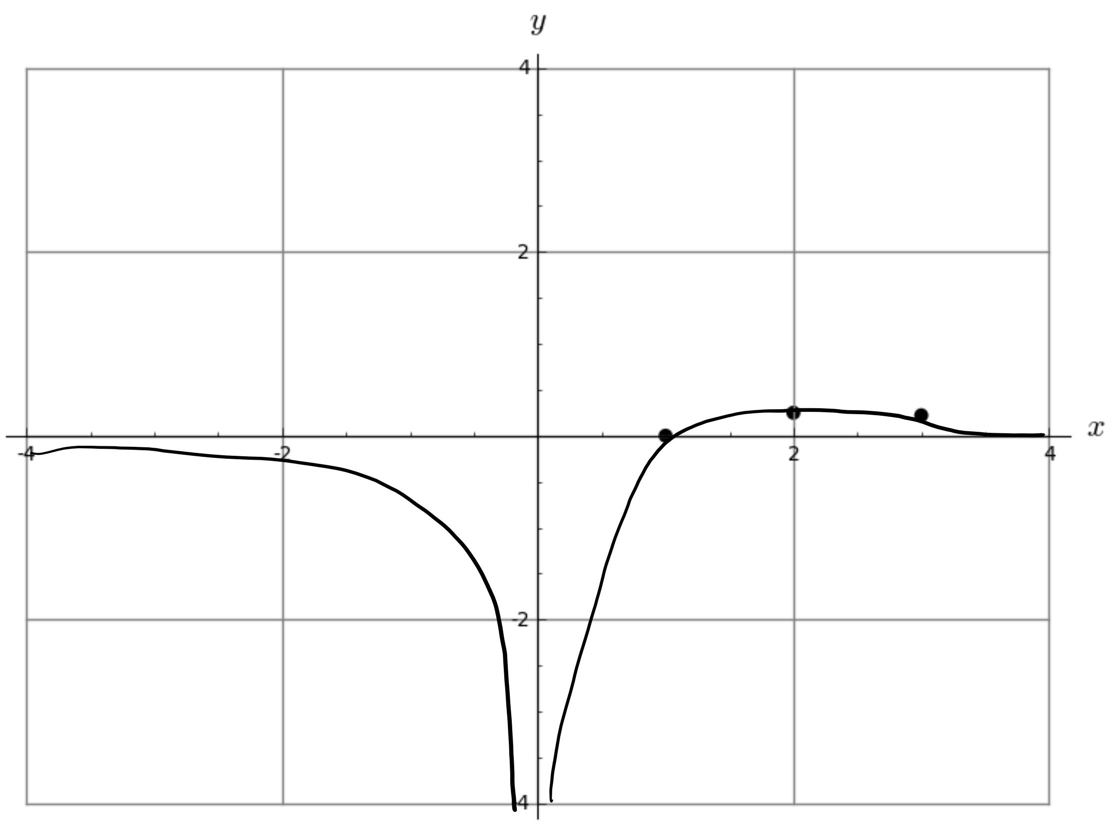
$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} \frac{2(x-2)}{x-3} = -\infty$$

Curve Sketching

12. You are given the following information about a function, f:

- x-intercept(s): $(1, 0)$,
- y-intercept: None,
- asymptotes: $x = 0$ and $y = 0$,
- critical point(s): $(2, 1/4)$,
- increasing: $(0, 2)$,
- decreasing: $(-\infty, 0) \cup (2, \infty)$,
- concave up: $(3, \infty)$,
- concave down: $(-\infty, 3)$, and
- $f(3) = 2/9$.

Use this information to sketch the curve.



Closed Interval Method and Optimization

13. Find the absolute maximum and minimum values of $f(x) = 12 + 4x - x^2$ on $[0, 5]$.

$f'(x) = 4 - 2x = 0 \Rightarrow x = 2.$

$f(x) = -(x^2 - 4x - 12) = -(x-6)(x+2)$

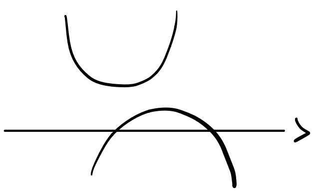
$f(0) = 12$

$f(2) = -(2-6)(2+2) = -(-4)(4) = 16$

$f(5) = -(5-6)(5+2) = -(-1)(7) = 7$

$\boxed{\text{Max: } (2,16) \quad \text{Min: } (5,7)}$

14. What is the minimum vertical distance between the parabolas $y = x^2 + 1$ and $y = x - x^2$?



$D(x) = (x^2+1) - (x - x^2)$

$= 2x^2 - x + 1$

$D'(x) = 4x - 1 = 0, \quad D''(x) = 4 > 0$

$\Rightarrow 4x = 1$

$\Rightarrow x = 1/4$ is a global minimum.

$D(1/4) = 2\left(\frac{1}{4}\right)^2 - \frac{1}{4} + 1$

$= 2\left(\frac{1}{16}\right) - \frac{1}{4} + 1$

$= \frac{1}{8} - \frac{2}{8} + \frac{8}{8}$

$= \boxed{7/8}$

Integration

Use the identity

$$\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$$

to compute the definite integral

$$\int_0^2 4x^3 \, \mathrm{d}x$$

by the **definition**.

$$\Delta x = \frac{2-0}{n} = \frac{2}{n}, \quad x_i = 0 + i\Delta x = \frac{2i}{n}$$

$$\int_0^2 x^3 dx = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{2i}{n}\right)^3 \left(\frac{2}{n}\right) = \lim_{n\to\infty} \frac{2^4}{n^4} \sum_{i=1}^{n} i^3$$

$$= \lim_{n\to\infty} \frac{16}{n^4} \frac{n^2(n+1)^2}{4} = \lim_{n\to\infty} \frac{4(n^2+2n+1)}{n^2}$$

$$= \lim_{n\to\infty} 4 + \frac{8}{n} + \frac{4}{n^2}$$

$$= 4 + 0 + 0 = 4$$

$$\int_0^2 4x^3 dx = 4\int_0^2 x^3 dx = 4(4) = \boxed{16}$$

Fundamental Theorem of Calculus

15. Evaluate the indefinite integral $\int \left(x^2 - x + \cos(x) - \sec^2(x)\right) \mathrm{d}x$.

$$\boxed{\frac{1}{3}x^3 - \frac{1}{2}x^2 + \sin(x) - \tan(x) + C.}$$

16. Given $f'(x) = 3x^2 + 2x + 1$ and $f(2) = 15$, find $f(x)$.

$$f(x) = \int f'(x)dx = 3\left(\frac{1}{3}x^3\right) + 2\left(\frac{1}{2}x^2\right) + x + C = x^3 + x^2 + x + C$$

$$15 = f(2) = 8 + 4 + 2 + C = 14 + C$$

$$\Rightarrow C = 15 - 14 = 1.$$

$$\boxed{f(x) = x^3 + x^2 + x + 1}$$

Substitution

17. Evaluate the indefinite integral $\int 2\sin(x)\cos(x)\,\mathrm{d}x$

$u = \sin(x)$

$du = \cos(x)dx$

$$\int 2\sin(x)\cos(x)dx = 2\int u\,du = 2\left(\frac{1}{2}u^2\right) + C$$

$$= \boxed{\sin^2(x) + C}$$

18. Evaluate the definite integral $\int_0^2 \frac{20x}{(x^2+1)^2}\,\mathrm{d}x$

$u = x^2+1$ $\quad$ $u(0) = 1$

$du = 2x dx$ $\quad$ $u(2) = 4+1 = 5$

$\Rightarrow 10du = 20x dx$

$$\int_0^2 \frac{20x}{(x^2+1)^2}dx = \int_1^5 \frac{10du}{u^2} = 10\int_1^5 u^{-2}du = 10\left(-1u^{-1}\right)\Big|_1^5$$

$$= -10\left(\frac{1}{5} - 1\right) = -2 + 10 = \boxed{8}$$