EXAM 3

BLAKE FARMAN

Lafayette College

Answer the questions below and submit them through Moodle before time expires.
It is advised, although not required, that you check your answers. All supporting work is required. Unsupported or otherwise mysterious answers will **not receive credit.**

Name: Solutions

Date: May 3, 2020.

Functions

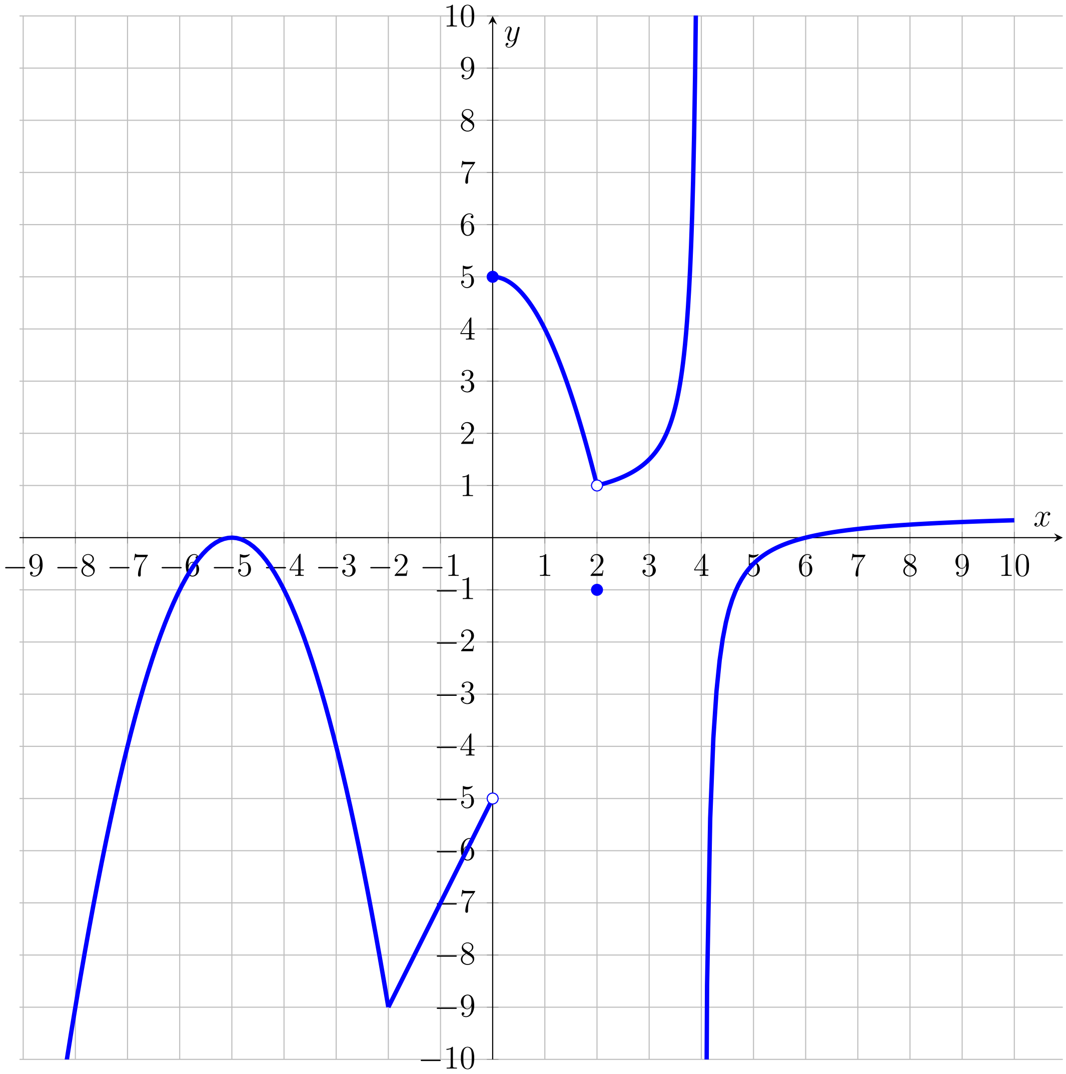


Figure 1. The graph of f.

1. Use Figure 1 to answer the following questions.

a. State all values a for which $\lim_{x\to a} f(x)$ does not exist. Justify your answers.

$a = 0$: $\lim_{x\to 0^-} f(x) = -5 \neq 5 = \lim_{x\to 0^+} f(x)$

$a = 4$: $\lim_{x\to 4^-} f(x) = \infty$ and $\lim_{x\to 4^+} f(x) = -\infty$.

b. State all values a for which f is discontinuous. Use the definition of continuity to justify your answers.

$a = 0$ and $a = 4$ because the limits do not exist.

$a = 2$ because

$\lim_{x\to 2} f(x) = 1 \neq -1 = f(2)$.

Derivatives

1. Use the **limit definition**

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

to compute the derivative of

$$f(x) = \frac{1}{x+1}.$$

$$\lim_{h\to 0} \frac{\frac{1}{x+h+1} - \frac{1}{x+1}}{h} = \lim_{h\to 0} \frac{1}{h}\left(\frac{x+1-(x+h+1)}{(x+1)(x+h+1)}\right)$$

$$= \lim_{h\to 0} \frac{-h}{h(x+1)(x+h+1)}$$

$$= \lim_{h\to 0} \frac{-1}{(x+1)(x+h+1)}$$

$$= \frac{-1}{(x+1)(x+0+1)}$$

$$= \boxed{\frac{-1}{(x+1)^2}}$$

Derivative Rules

1. Consider the following differentiable functions, f and g, satisfying

- $f(3) = -1$ and $f'(3) = 2$,
- $g(3) = 1$ and $g'(3) = 3$.

Compute the line tangent to the function

$$h(x) = f(x) + x^2 - g(x)$$

at $x = 3$.

$$h(3) = f(3) + 3^2 - g(3)$$

$$= -1 + 9 - 1$$

$$= 7$$

$$h'(x) = f'(x) + 2x - g'(x)$$

$$h'(3) = f'(3) + 2(3) - g'(3)$$

$$= 2 + 6 - 3$$

$$= 8 - 3$$

$$= 5$$

$\boxed{y - 7 = 5(x-2)}$ or $\boxed{y = 5x - 3}$

Product and Quotient Rules

Assume that f is a differentiable function satisfying

$$f\left(\frac{\pi}{2}\right) = 2 \quad \text{and} \quad f'\left(\frac{\pi}{2}\right) = 4.$$

Use this function to compute $g'(\pi/2)$ and $h'(\pi/2)$ below.

1. $g(x) = \cos(x)f(x)$

$$g'(x) = -\sin(x)f(x) + \cos(x)f'(x)$$

$$g'\left(\frac{\pi}{2}\right) = -\sin\left(\frac{\pi}{2}\right)f\left(\frac{\pi}{2}\right) + \cos\left(\frac{\pi}{2}\right)f'\left(\frac{\pi}{2}\right)$$

$$= -(1)(2) + 0(4)$$

$$= \boxed{-2}$$

2. $h(x) = \dfrac{\cos(x)}{f(x)}$.

$$h'(x) = \frac{-\sin(x)f(x) - \cos(x)f'(x)}{f(x)^2}$$

$$h'\left(\frac{\pi}{2}\right) = \frac{-\sin\left(\frac{\pi}{2}\right)f\left(\frac{\pi}{2}\right) - \cos\left(\frac{\pi}{2}\right)f'\left(\frac{\pi}{2}\right)}{f\left(\frac{\pi}{2}\right)^2}$$

$$= \frac{-(1)(2) - 0(4)}{(2)^2}$$

$$= \frac{-2}{4}$$

$$= \boxed{\frac{-1}{2}}$$

Chain Rule

1. Assume that f is a differentiable function satisfying $f'(3) = 4$, $f(3) = 2$, and let $g(x) = \sqrt{f(x)+2}$. Compute $g'(3)$.

$$g'(x) = \frac{\frac{d}{dx}(f(x)+2)}{2\sqrt{f(x)+2}}$$

$$= \frac{f'(x)}{2\sqrt{f(x)+2}}$$

$$g'(3) = \frac{f'(3)}{2\sqrt{f(3)+2}}$$

$$= \frac{4}{2\sqrt{2+2}}$$

$$= \frac{2}{\sqrt{4}}$$

$$= \frac{2}{2}$$

$$= \boxed{1}$$

Implicit Differentiation and Related Rates

1. If a snowball melts so that its surface area, $S = 4\pi r^2$, decreases at a rate of 1 cm^2/min, find the rate at which the diameter decreases when the diameter is 10 cm.

$$d = 2r \Rightarrow d^2 = 4r^2$$

$$S = 4\pi r^2 = \pi 4r^2 = \pi d^2$$

$$S' = 2\pi d d'$$

$$\Rightarrow d' = \frac{S'}{2\pi d}$$

$$= \frac{-1}{2\pi(10)}$$

$$= \boxed{\frac{-1}{20\pi} \text{ cm/s}}$$

Derivatives and Shape

Use the function

$$f(x) = -2x^3 + 12x^2 - 18x + 4$$

to answer the following questions.

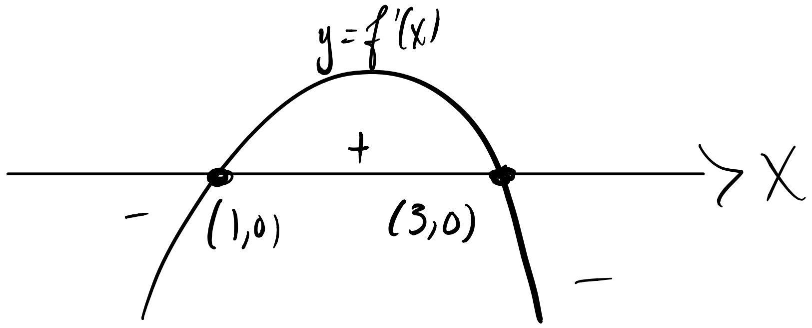
1. Find the intervals where the function is increasing and decreasing. Write NONE if not applicable.

Increasing: $(1, 3)$

Decreasing: $(-\infty, 1) \cup (3, \infty)$

$$f(x) = -2x^3 + 12x^2 - 18x + 4$$

$$f'(x) = -6x^2 + 24x - 18 = -6(x^2 - 4x + 3) = -6(x-1)(x-3)$$



2. State the local maximum and local minimum value(s). Write NONE if not applicable.

Local maximum value(s): $(3, 4)$

Local minimum value(s): $(1, -4)$

$$f(3) = -2(3)^3 + 12(3)^2 - 18(3) + 4$$
$$= 9(-6 + 12 - 6) + 4$$
$$= 4$$

$$f(1) = -2 + 12 - 18 + 4$$
$$= -20 + 16$$
$$= -4.$$

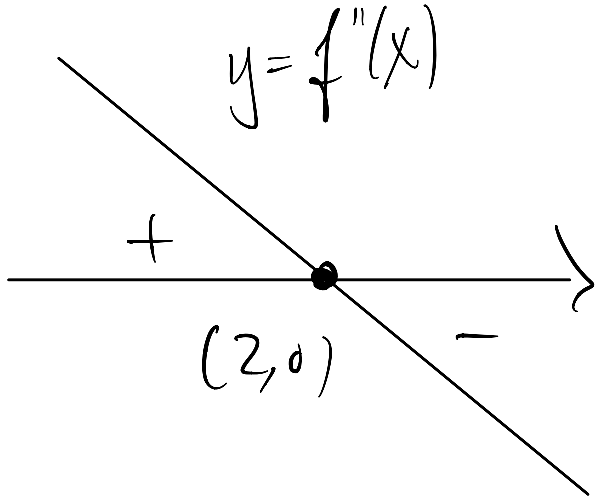
3. Find the intervals on which the function is concave up and concave down. State the inflection points. Write NONE if not applicable.

Concave Up: $(-\infty, 2)$

Concave Down: $(2, \infty)$

Inflection Points: $(2, 16)$

$f''(x) = -12x + 24 = -12(x-2)$



$f(2) = -2(8) + 12(4) - 18(2) + 4$
$= 4(-8 + 12 - 9) + 4$
$= 4(-17 + 13) = 4(-4) = -16$

Asymptotes

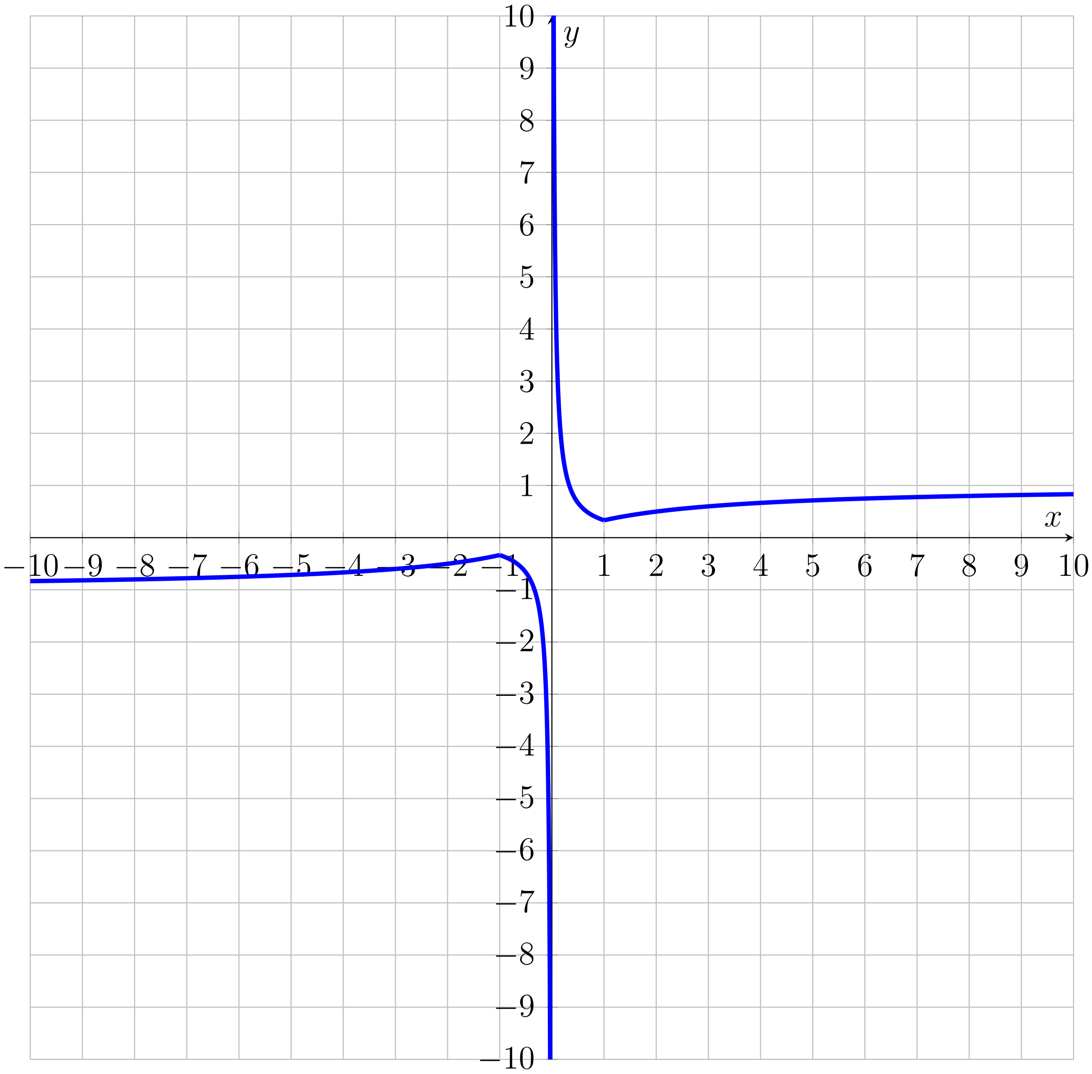


Figure 2. The graph of f

Use Figure 2 to answer the following questions.

1. List any vertical asymptotes of the function f. Justify your answers using limits.

$\underline{x=0}$: $\lim_{x\to 0^-} f(x) = -\infty$ and $\lim_{x\to 0^+} f(x) = \infty$

2. List any horizontal asymptotes of the function f. Justify your answers using limits.

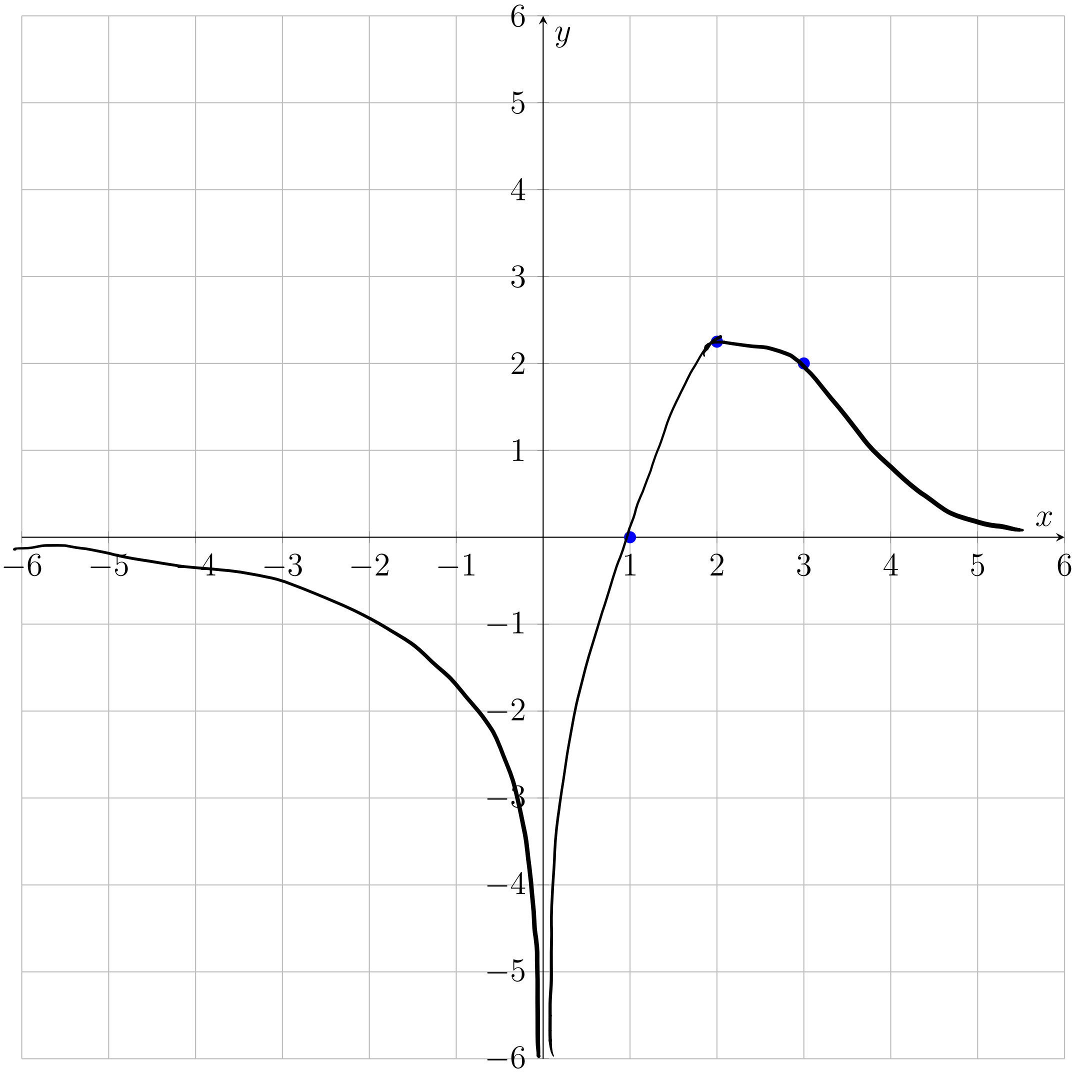
$\underline{y=1}$: $\lim_{x\to\infty} f(x) = 1$ $\underline{y=-1}$: $\lim_{x\to -\infty} f(x) = -1$.

Curve Sketching

1. You are given the following information about a function, f:

- x-intercept(s): $(1, 0)$,
- y-intercept: None,
- asymptotes: $x = 0$ and $y = 0$,
- critical point(s): $(2, 9/4)$,
- increasing: $(0, 2)$,
- decreasing: $(-\infty, 0) \cup (2, \infty)$,
- concave up: $(3, \infty)$,
- concave down: $(-\infty, 3)$, and
- $f(3) = 2$.

Use this information to sketch the curve. From left to right, the points $(1, 0)$, $(2, 9/4)$, and $(3, 2)$have been plotted for you.



Closed Interval Method and Optimization

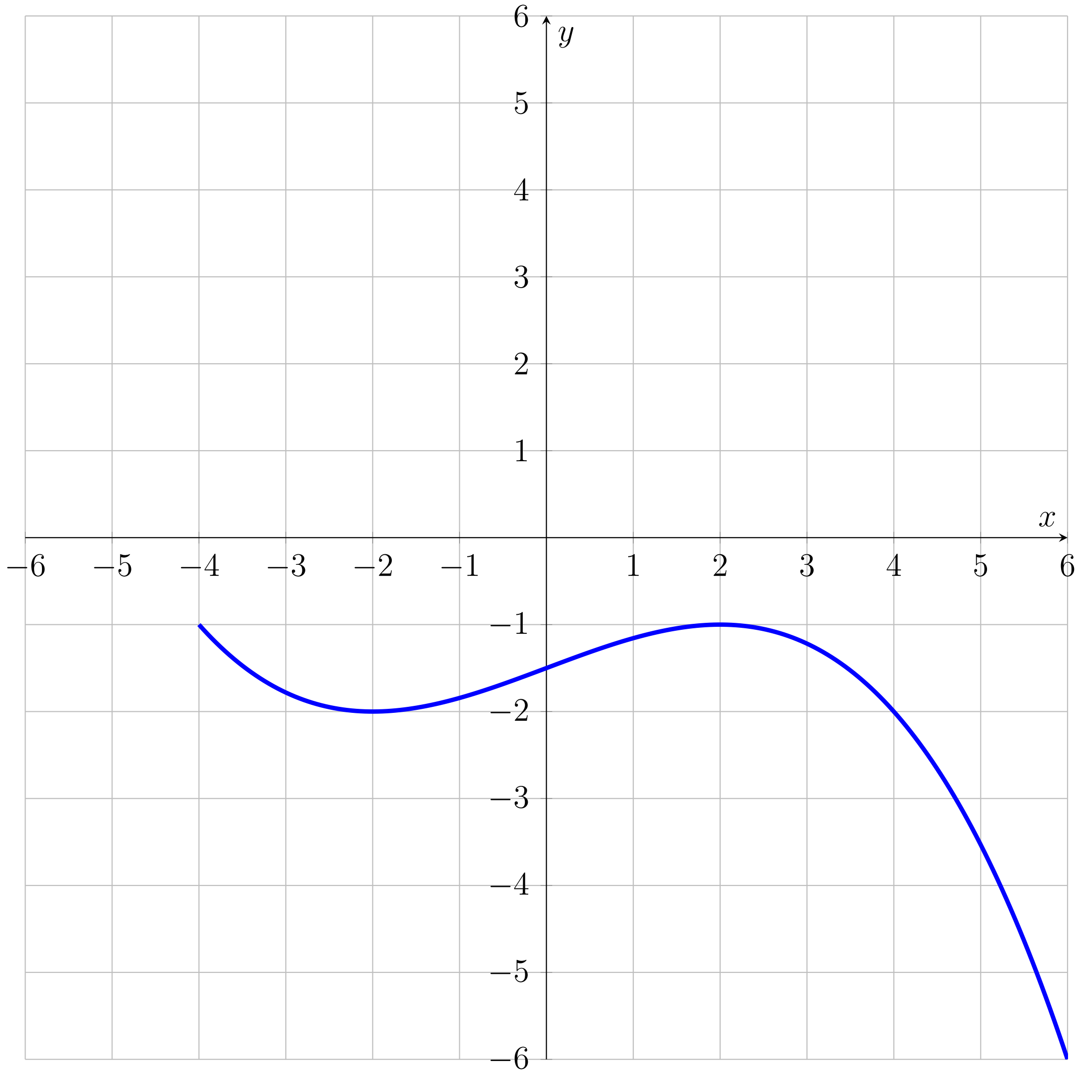


Figure 3. The graph of f

1. Use Figure 3 to find the absolute maximum and minimum values of f on the interval $[-4, 6]$. List your solutions as an (x, y) pair.

Maximum: (-4,-1) and (2,-1)

Minimum: (6,-6)

2. Karen runs the ACME Widget Company, which sells Widgets for \$21 each. Karen observes that she can model the cost of manufacturing x thousand Widgets each month by the function

$$C(x) = \frac{1}{3}x^3 - 2x^2 + 100$$

in thousands of dollars.

Knowing that the profit from selling x thousand Widgets is given by

$$P(x) = 21x - C(x)$$

and having taken Math 161, Karen decides that she can maximize her profits by minimizing the costs. She observes that for $x > 0$

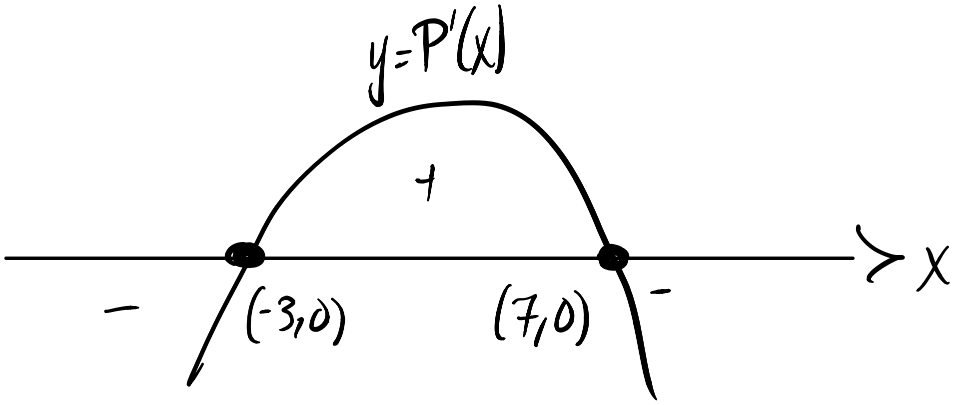
$$C'(x) = x^2 - 4x = x(x-4) = 0 \iff x = 4$$

so she concludes that she should manufacture 4,000 widgets per month.

Six months later, having sold every Widget manufactured, Karen has lost \$32,000. What was her mistake? How many Widgets should she have chosen to produce each month?

The minimum of the cost function is <u>not</u> a maximum of the profit function.

$$P'(x) = 21 - C'(x) = 21 - x^2 + 4x = -(x^2 - 4x + 21) = -(x-7)(x+3)$$



We can see the maximum occurs at $x=7$ by the First Derivative Test, so she should produce 7,000 widgets each month.

Integration

1. Use **Right Endpoints** and the formulas

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$$

to compute the definite integral

$$\int_1^2 2x^2 \, \mathrm{d}x$$

$$\Delta x = \frac{2-1}{n} = \frac{1}{n} \implies x_i = 1 + \frac{i}{n}$$

$$f(x_i)\Delta x = 2\left(1+\frac{i}{n}\right)^2\left(\frac{1}{n}\right) = 2\left(1+\frac{2}{n}i + \frac{1}{n^2}i^2\right)\left(\frac{1}{n}\right)$$

$$= \frac{2}{n} + \frac{4}{n^2}i + \frac{2}{n^3}i^2$$

$$R_n = \sum_{i=1}^{n} f(x_i)\Delta x = \sum_{i=1}^{n} \frac{2}{n} + \sum_{i=1}^{n} \frac{4}{n^2}i + \sum_{i=1}^{n} \frac{2}{n^3}i^2$$

$$= \frac{2}{n}\sum_{i=1}^{n} 1 + \frac{4}{n^2}\sum_{i=1}^{n} i + \frac{2}{n^3}\sum_{i=1}^{n} i^2$$

$$= \frac{2}{n}(n) + \frac{4}{n^2}\,\frac{n(n+1)}{2} + \frac{2}{n^3}\,\frac{n(n+1)(2n+1)}{6}$$

$$= 2 + \frac{2(n^2+n)}{n^2} + \frac{2n^2+3n+1}{3n^2}$$

$$= 2 + 2 + \frac{2}{n} + \frac{2}{3} + \frac{1}{n} + \frac{1}{3n^2}$$

$$= \frac{14}{3} + \frac{3}{n} + \frac{1}{3n^2}$$

$$\int_1^2 2x^2\,dx = \lim_{n\to\infty} R_n = \lim_{n\to\infty}\left[\frac{14}{3} + \frac{3}{n} + \frac{1}{3n^2}\right] = \frac{14}{3} + 0 + 0 = \boxed{\frac{14}{3}}$$

Check: $\int_1^2 2x^2\,dx = \frac{2}{3}x^3\Big|_1^2 = \frac{2}{3}(8-1) = \frac{2}{3}(7) = \frac{14}{3}$ ✓

Fundamental Theorem of Calculus

1. Consider the function $F(x)$ in Figure 4.

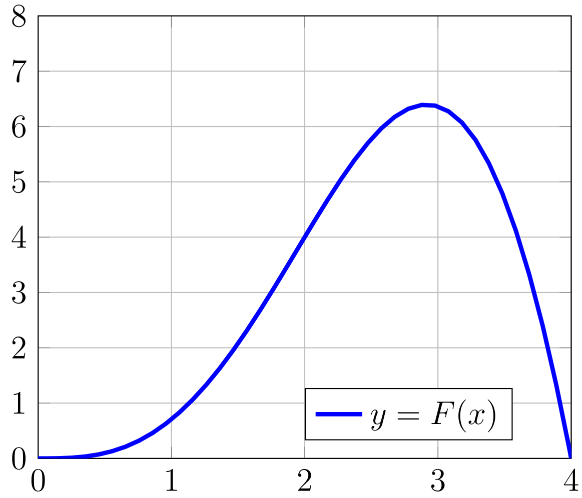


Figure 4. The graph of $y = F(x)$

Assume that f is a continuous function such that $f(x) = F'(x)$. Compute

$$\int_0^2 f(x)\,\mathrm{d}x.$$

$$\int_0^2 f(x)dx = F(2) - F(0) = 4 - 0 = 4.$$

2. Assume that $f'(x) = 3x^2 + 2x + 1$ and $f(2) = 15$. Find $f(x)$.

$$f(x) = \int f'(x)dx = 3\int x^2 dx + 2\int x dx + \int dx$$

$$= 3\left(\frac{1}{3}\right)x^3 + 2\left(\frac{1}{2}x^2\right) + x + C$$

$$= x^3 + x^2 + x + C$$

$$f(2) = 15 = 8 + 4 + 2 + C = 14 + C \Rightarrow C = 1.$$

$$\boxed{f(x) = x^3 + x^2 + x + 1}$$

Substitution

1. Evaluate the indefinite integral

$$\int 2\sin(x)\cos(x)\,\mathrm{d}x.$$

$u = \sin(x)$
$du = \cos(x)$

$$\int 2\sin(x)\cos(x)dx = 2\int u\,du = 2\left(\tfrac{1}{2}u^2\right) + C = u^2 + C = \boxed{\sin^2(x) + C.}$$

or

$u = \cos(x)dx$
$du = -\sin(x)dx$

$$\int 2\sin(x)\cos(x)dx = -2\int u\,du = -2\left(\tfrac{1}{2}u^2\right) + C = \boxed{-\cos^2(x) + C}$$

or $\int 2\sin(x)\cos(x)dx = \int \sin(2x)dx$

$u = 2x$
$du = 2dx \Rightarrow \frac{1}{2}du = dx$

$$\int \sin(2x)dx = \int \tfrac{1}{2}\sin(u)\,du = \tfrac{-1}{2}\cos(u) + C = \boxed{\tfrac{-1}{2}\cos(2x) + C.}$$

2. Assume that f is a differentiable function such that $f(0) = 0$ and $f(1) = \pi/2$. Evaluate the definite integral

$$\int_0^1 f'(x)\sin\left(2f(x)\right)\mathrm{d}x.$$

$u = 2f(x)$
$du = 2f'(x)\,dx$
$\Rightarrow \frac{1}{2}du = f'(x)dx$

$u(0) = 2f(0) = 2(0) = 0$
$u(1) = 2f(1) = 2(\pi/2) = \pi$

$$\int_0^1 f'(x)\sin(2f(x))\,dx = \int_0^{\pi} \sin(u)\left(\tfrac{1}{2}du\right) = \tfrac{-1}{2}\cos(u)\Big|_0^{\pi}$$

$$= \frac{-1}{2}\left[\cos(\pi) - \cos(0)\right]$$

$$= \frac{-1}{2}\left[-1-1\right] = \frac{-1}{2}(-2) = \boxed{1}$$