EXAM 2

BLAKE FARMAN

Lafayette College

> Answer the questions in the spaces provided on the question sheets and turn them in at the end of the exam period.
> It is advised, although not required, that you check your answers. All supporting work is required. Unsupported or otherwise mysterious answers will **not receive credit.**
> You may **not** use a calculator or any other electronic device, including cell phones, smart watches, etc.

Name: Solutions

Functions

1. Find the value of b that makes the function

$$f(x) = \begin{cases} \dfrac{x^2-25}{x+5} & \text{if } -5 < x \\ 5x+b & \text{if } x \leq -5 \end{cases}$$

a continuous function.

$$\lim_{x\to -5^+} f(x) = \lim_{x\to -5^+} \frac{x^2-25}{x+5} = \lim_{x\to -5^+} \frac{(x+5)(x-5)}{x+5} = \lim_{x\to -5^+} x-5 = -5-5 = -10$$

$$\lim_{x\to -5^-} f(x) = f(-5) = 5(-5)+b = -25+b = -10$$

$$\Rightarrow b = -10+25 = \boxed{15}$$

Date: July 24, 2019.

Derivatives

2. Use the **limit definition**

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

to compute the derivative of

$$f(x) = \frac{1}{x}.$$

$$\lim_{h\to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} = \lim_{h\to 0} \frac{1}{h}\left(\frac{x}{(x+h)x} - \frac{x+h}{x(x+h)}\right)$$

$$= \lim_{h\to 0} \frac{1}{h}\left(\frac{x - x - h}{x(x+h)}\right)$$

$$= \lim_{h\to 0} \frac{-h}{hx(x+h)}$$

$$= \lim_{h\to 0} \frac{-1}{x(x+h)}$$

$$= \frac{-1}{x(x+0)}$$

$$= \boxed{\frac{-1}{x^2}}.$$

Derivative Rules

3. Compute the line tangent to

$$f(x) = \tan(x)$$

at the point $(\pi/4, 1)$.

$f'(x) = \sec^2(x)$

$f'\left(\frac{\pi}{4}\right) = \sec^2(\pi/4) = \left(\frac{2}{\sqrt{2}}\right)^2 = \frac{4}{2} = 2$

$\boxed{y - 1 = 2(x - \pi/4)}$

Product and Quotient Rules

4. Compute

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[x^2\sec(x)\right].$$

$$\frac{d}{dx}\left(x^2\sec(x)\right) = \frac{d}{dx}(x^2)\sec(x) + x^2\frac{d}{dx}\left(\sec(x)\right)$$

$$= \boxed{2x\sec(x) + x^2\sec(x)\tan(x)}$$

5. Compute

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\cos(x)}{\sin^2(x)}\right].$$

$$\frac{d}{dx}\frac{\cos(x)}{\sin^2(x)} = \frac{d/dx\left(\cos(x)\right)\sin^2(x) - \cos(x)\frac{d}{dx}\sin^2(x)}{\left[\sin^2(x)\right]^2}$$

$$= \frac{-\sin(x)\sin^2(x) - \cos(x)\left(2\sin(x)\frac{d}{dx}\sin(x)\right)}{\sin^4(x)}$$

$$= \frac{-\sin^3(x) - 2\cos(x)\sin(x)\cos(x)}{\sin^4(x)}$$

$$= \boxed{\frac{-\sin^2(x) - 2\cos^2(x)}{\sin^4(x)}}$$

Chain Rule

6. Compute

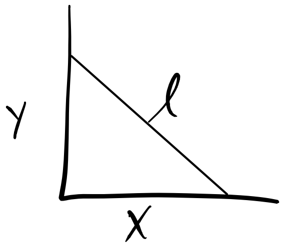
$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\cos\left(2x^2+5x\right)\right].$$

$$\frac{d}{dx}\cos(2x^2+5x) = -\sin(2x^2+5x)\frac{d}{dx}(2x^2+5x)$$

$$= \boxed{-\sin(2x^2+5x)(4x+5)}$$

Implicit Differentiation and Related Rates

7. The top of a ladder slides down a vertical wall at a rate of 3 meters/second. At the moment when the top of the ladder is 4 meters from the ground, it slides away from the wall at a rate of 4 meters/second. How long is the ladder?



$$\ell^2 = y^2 + x^2$$

$$2\ell \frac{d\ell}{dt} = 0 = 2y\frac{dy}{dt} + 2x\frac{dx}{dt}$$

$$\Rightarrow 0 = y\frac{dy}{dt} + x\frac{dx}{dt}$$

$$= 4(-3) + x(3)$$

$$= -12 + 3x$$

$$\Rightarrow 3x = 12$$

$$\Rightarrow x = 12/3 = 4$$

So $\ell = \sqrt{3^2 + 4^2} = \sqrt{25} = \boxed{5m}$

Derivatives and Shape

Use the function

$$f(x) = x^3 + 9x^2$$

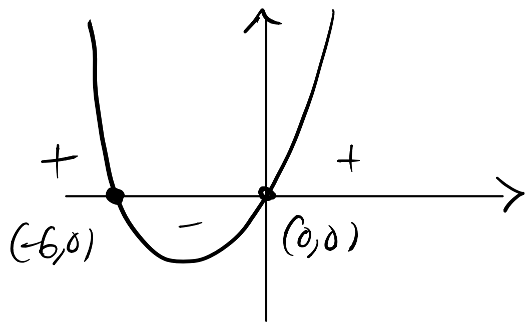
to answer the following questions.

8. Find the intervals where the function is increasing and decreasing. Write NONE if not applicable.

Increasing: $(-\infty, -6) \cup (0, \infty)$

Decreasing: $(-6, 0)$

$$f'(x) = 3x^2 + 18x = 3x(x+6) = 0 \Rightarrow x=0 \text{ or } x=-6$$



9. State the local maximum and local minimum value(s). Write NONE if not applicable.

Local maximum value(s): $(-6, -108)$

Local minimum value(s): $(0, 0)$

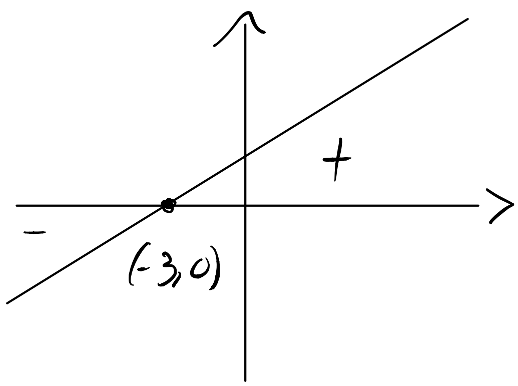
10. Find the intervals on which the function is concave up and concave down. State the inflection points. Write NONE if not applicable.

Concave Up: $(-\infty, -3)$

Concave Down: $(-3, \infty)$

Inflection Points: $(-3, 0)$

$$f''(x) = 6x + 18 = 6(x+3) = 0 \Rightarrow x = -3$$



Asymptotes

11. Find the asymptotes of

$$f(x) = \frac{3x^2}{x^2 - 4}$$

Write NONE if there are none.

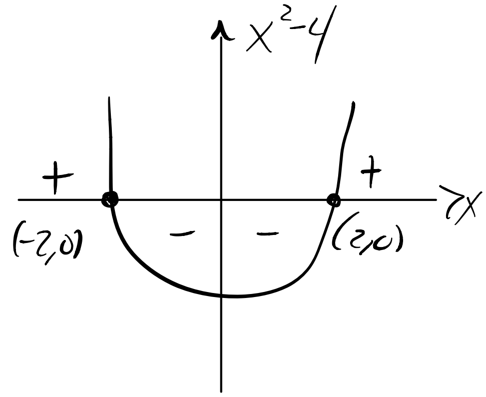
Horizontal: $y = 3$

Vertical: $x = \pm 2$

$$\lim_{x \to \infty} \frac{3x^2}{x^2-4} = \lim_{x \to \infty} \frac{x^2}{x^2}\left(\frac{3}{1-4/x^2}\right) = \lim_{x \to \infty} \frac{3}{1-4/x^2} = \frac{3}{1-0} = 3$$

Since $3x^2 \geq 0$

$$\lim_{x \to 2^+} f(x) = \infty, \quad \lim_{x \to 2^-} f(x) = -\infty$$



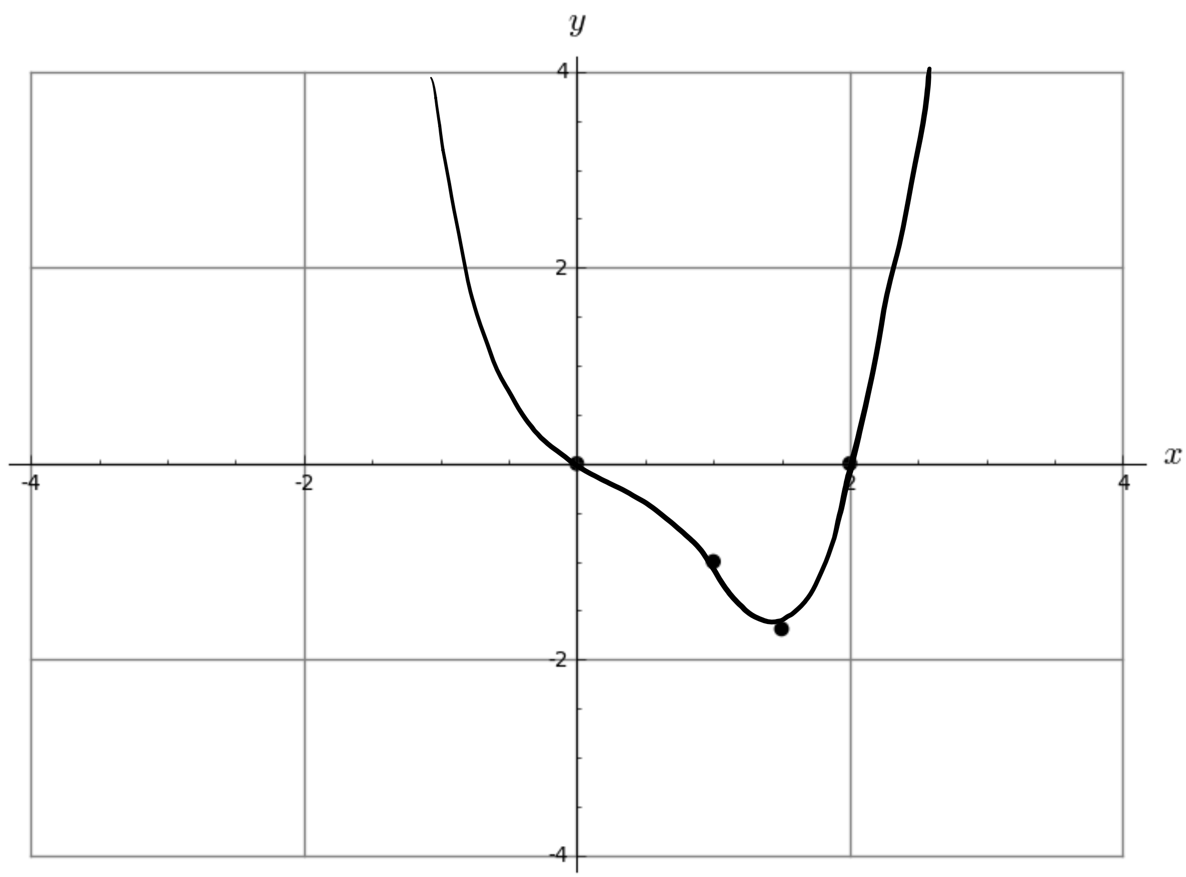
$$\lim_{x \to -2^+} f(x) = -\infty, \quad \lim_{x \to -2^-} f(x) = \infty$$

Curve Sketching

12. You are given the following information about a function, f:

- x-intercepts: $(0,0)$ and $(2,0)$,
- y-intercepts: $(0,0)$,
- asymptotes: none,
- critical points: $(0,0)$ and $\left(\frac{3}{2}, -\frac{27}{16}\right)$,
- increasing: $\left(\frac{3}{2}, \infty\right)$,
- decreasing: $(-\infty, 3/2)$,
- concave up: $(-\infty, 0) \cup (1, \infty)$,
- concave down: $(0,1)$, and
- $f(1) = -1$.

Use this information to sketch the curve.



Closed Interval Method and Optimization

13. Find the absolute maximum and minimum values of $f(x) = x^3 - 6x^2 + 9x$ on $[-1, 4]$.

$$f(x) = x(x^2 - 6x + 9) = x(x-3)^2$$

$$f'(x) = 3x^2 - 12x + 9 = 3(x^2 - 4x + 3) = 3(x-1)(x-3) = 0$$

$$\Rightarrow x = 1 \text{ or } x = 3$$

$$f(-1) = -1(-1-3)^2 = -16 \qquad f(4) = 4(4-3)^2 = 4$$

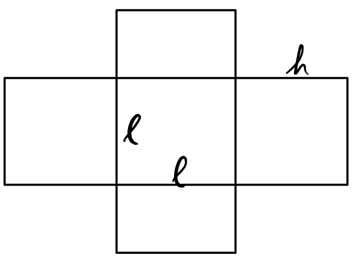
$$f(1) = 1(1-3)^2 = 4$$

$$f(3) = 3(3-3)^2 = 0$$

<u>Max</u>: $(1,4)$ & $(4,4)$

<u>Min</u>: $(-1,-16)$

14. A box with a square base and no top must have a volume of $4\,\text{cm}^3$. Find the dimensions of the box that minimize the amount of material used.



$$V = \ell^2 h = 4 \Rightarrow h = 4/\ell^2$$

$$S = \ell^2 + 4\ell h = \ell^2 + 4\ell\left(\frac{4}{\ell^2}\right)$$

$$= \ell^2 + \frac{16}{\ell} = \ell^2 + 16\ell^{-1}$$

$$S' = 2\ell - 16\ell^{-2} = 0$$

$$\Rightarrow 2\ell = \frac{16}{\ell^2} \Rightarrow \ell = \frac{8}{\ell^2}$$

$$\Rightarrow \ell^3 = 8 \Rightarrow \ell = 2$$

$$h = 1$$

When $\ell > 0$

$$S'' = 2 + 32\ell^{-3}$$

$$= 2 + \frac{\ell^3}{32} > 0$$

$\Rightarrow \ell = 2$ a minimum.