L'HÔPITAL'S RULE

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Name: Solutions

1. Compute

$$\lim_{x\to 1} \frac{x^3-2x^2+1}{x^3-1}$$

in two ways: with and without using L'Hôpital's Rule.

$$\lim_{x\to 1} \frac{x^3-2x^2+1}{x^3-1} \overset{L'H}{=} \lim_{x\to 1} \frac{3x^2-4x}{3x^2} = \frac{3-4}{3} = \boxed{\frac{-1}{3}}$$

Without L'Hôpital: factor an $x-1$ out of each term

$$x^3-1 = (x-1)(x^2+x+1)$$

$$\begin{array}{r l}
 & x^2 - x - 1 \\
x-1 \,\big) & x^3 - 2x^2 + 1 \\
 & -x^3 + x^2 \\
 & \quad -x^2 + 1 \\
 & \quad +x^2 - x \\
 & \qquad -x + 1 \\
 & \qquad +x - 1 \\
 & \qquad\quad 0
\end{array}$$

$$\Rightarrow \lim_{x\to 1} \frac{x^3-2x^2+1}{x^3-1} = \lim_{x\to 1} \frac{(x-1)(x^2-x-1)}{(x-1)(x^2+x+1)}$$

$$= \lim_{x\to 1} \frac{x^2-x-1}{x^2+x+1}$$

$$= \frac{1-1-1}{1+1+1}$$

$$= \boxed{\frac{-1}{3}}$$

Evaluate the following limits.

$\frac{0}{0}$ **2.** $\displaystyle\lim_{x\to\pi}\frac{\sin(3x)}{x-\pi} \overset{L'H}{=} \lim_{x\to\pi}\frac{3\cos(3x)}{1}$

$= 3(-1)$

$= \boxed{-3}$

3. $\displaystyle\lim_{t\to 0}\frac{e^{2t}-1}{e^t} = \frac{1-1}{1}$

$= \boxed{0}$

$\frac{0}{0}$ **4.** $\displaystyle\lim_{\theta\to 0}\frac{\arctan(\theta)}{2\theta} \overset{L'H}{=} \lim_{\theta\to 0}\frac{\left(\frac{1}{1+x^2}\right)}{2}$

$= \frac{1}{2}\left(\frac{1}{1+0}\right)$

$= \boxed{\frac{1}{2}}$

5. $\lim_{x\to\infty} \frac{e^{-x}}{1+\ln(x)}$ $= \lim_{x\to\infty} \frac{1}{e^{x}(1+\ln(x))}$

$= \boxed{0}$

$\frac{\infty}{\infty}$ **6.** $\lim_{x\to\infty} \frac{(\ln(x))^2}{x}$ $\overset{L'H}{=} \lim_{x\to\infty} \frac{2\ln(x)\left(\frac{1}{x}\right)}{1}$

$= \lim_{x\to\infty} \frac{2\ln(x)}{x}$

$\overset{L'H}{=} \lim_{x\to\infty} \frac{2(1/x)}{1} = \boxed{0}$

7. $\lim_{u\to\infty} \frac{\sqrt{u^2+1}}{u}$ $= \lim_{u\to\infty} \sqrt{\frac{u^2+1}{u^2}}$

$= \sqrt{1}$

$= \boxed{1}$

$\infty - \infty$ **8.** $\lim_{x\to\infty} (x - \ln(x)) = \lim_{x\to\infty} \left[\ln(e^x) - \ln(x)\right]$

$$= \lim_{x\to\infty} \ln\left(\frac{e^x}{x}\right) = \infty$$

Because $\lim_{x\to\infty} \frac{e^x}{x} \overset{L'H}{=} \lim_{x\to\infty} e^x = \infty.$

$\infty - \infty$ **9.** $\lim_{x\to 1} \left(\frac{x}{x-1} - \frac{1}{\ln(x)}\right) = \lim_{x\to 1} \frac{x\ln(x) - (x-1)}{(x-1)\ln(x)}$

$$\overset{L'H}{=} \lim_{x\to 1} \frac{\ln(x) + 1 - 1}{\ln(x) + \frac{x-1}{x}}$$

$$= \lim_{x\to 1} \frac{\ln(x)}{\ln(x) + 1 - 1/x}$$

$$\overset{L'H}{=} \lim_{x\to 1} \frac{\frac{1}{x}}{\frac{1}{x} + \frac{1}{x^2}} = \lim_{x\to 1} \frac{\frac{1}{x}}{\frac{x+1}{x^2}} = \lim_{x\to 1} \frac{x^2}{x(x+1)} = \boxed{\frac{1}{2}}$$

$\infty - \infty$ **10.** $\lim_{x\to 1^+} \left[\ln(x^7 - 1) - \ln(x^5 - 1)\right] = \lim_{x\to 1^+} \ln\left(\frac{x^7-1}{x^5-1}\right)$

$$= \ln\left(\lim_{x\to 1^+} \frac{x^7-1}{x^5-1}\right) \overset{L'H}{=} \ln\left(\lim_{x\to 1} \frac{7x^6}{5x^4}\right) = \boxed{\ln(7/5)}$$

0^0 **11.** $\lim_{x\to 0^+} x^{\sqrt{x}}$

$$\lim_{x\to 0^+} \ln\left(x^{\sqrt{x}}\right) = \lim_{x\to 0^+} \sqrt{x}\ln(x)$$

$$= \lim_{x\to 0^+} \frac{\ln(x)}{x^{-1/2}}$$

$$\overset{L'H}{=} \lim_{x\to 0^+} \frac{x^{-1}}{-\frac{1}{2}x^{-3/2}}$$

$$= \lim_{x\to 0^+} -2x^{-1+3/2} = \lim_{x\to 0^+} -2\sqrt{x} = 0$$

$$\Rightarrow \lim_{x\to 0^+} x^{\sqrt{x}} = e^0 = \boxed{1}$$

1^∞ **12.** $\lim_{x\to 0} (1-2x)^{1/x}$

$$\lim_{x\to 0} \ln\left((1-2x)^{1/x}\right) = \lim_{x\to 0} \frac{\ln(1-2x)}{x}$$

$$\overset{L'H}{=} \lim_{x\to 0} \frac{\frac{-2}{1-2x}}{1} = \frac{-2}{1} = -2$$

$$\Rightarrow \lim_{x\to 0} (1-2x)^{1/x} = \boxed{e^{-2} = 1/e^2}$$

∞^0 **13.** $\lim_{x\to\infty} x^{1/x}$

$$\lim_{x\to\infty} \ln\left(x^{1/x}\right) = \lim_{x\to\infty} \frac{\ln(x)}{x} \overset{L'H}{=} \lim_{x\to\infty} \frac{1/x}{1} = 0$$

$$\Rightarrow \lim_{x\to\infty} x^{1/x} = e^0 = \boxed{1}$$