INTEGRAL TEST AND ESTIMATES OF SUMS

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Name: Solutions

Use the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ to answer the following questions.

1. Show that the function $f(x) = 1/x^2$ satisfies the hypotheses of the Integral Test. For n a fixed integer, compute the improper integral

$$\int_n^{\infty} \frac{1}{x^2}\,\mathrm{d}x.$$

Use this to conclude that the series $\sum_{n=1}^{\infty} 1/n^2$ converges.

$$\int_n^{\infty} \frac{1}{x^2}dx = \lim_{t\to\infty} \int_n^t \frac{1}{x^2}dx$$

$$= \lim_{t\to\infty} -\frac{1}{x}\Big|_n^t$$

$$= \lim_{t\to\infty} -\left(\frac{1}{t} - \frac{1}{n}\right)$$

$$= -\left(0 - \frac{1}{n}\right)$$

$$= \frac{1}{n}$$

Since $\int_1^{\infty} \frac{1}{x^2}dx = \frac{1}{1} = 1$, $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by the integral test.

2. Use a calculator to find

$$s_{10} = \sum_{n=1}^{10} \frac{1}{n^2}.$$

Use the intequality

$$\int_{11}^{\infty} \frac{1}{x^2}\,\mathrm{d}x \le R_{10} \le \int_{10}^{\infty} \frac{1}{x^2}\,\mathrm{d}x$$

to determine how good this estimate to the sum of the series is.

$$S_{10} = \sum_{n=1}^{10} \frac{1}{n^2} \approx 1.55$$

$$\frac{1}{11} = \int_{11}^{\infty} \frac{1}{x^2}dx \le R_{10} \le \int_{10}^{\infty} \frac{1}{x^2}dx = \frac{1}{10}$$

$\Rightarrow$ S_{10} is correct to at least one decimal place.

3. Use the inequality

$$s_{10} + \int_{11}^{\infty} \frac{1}{x^2}\,\mathrm{d}x \le \sum_{n=1}^{\infty} \frac{1}{n^2} \le s_{10} + \int_{10}^{\infty} \frac{1}{x^2}\,\mathrm{d}x$$

to find an open interval containing the number s. Compute the midpoint of this interval. Is the midpoint a better or worse approximation to the sum of the series than you found in Problem 2? Why or why not?

$$S_{10} + 1/11 \le S \le S_{10} + \frac{1}{10}$$

The midpoint of this interval is

$$\frac{\left(S_{10} + \frac{1}{11}\right) + \left(S_{10} + \frac{1}{10}\right)}{2} = S_{10} + \frac{\frac{10}{110} + \frac{11}{110}}{2}$$

$$= S_{10} + \frac{21}{220}$$

$$\approx 1.6452$$

This is a better approximation because the error is at most half the of the interval containing S:

$$\frac{\frac{1}{10} - \frac{1}{11}}{2} = \frac{11-10}{2(110)} = \frac{1}{220}$$

4. It is known that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Use this to compare your estimates from Problems 2 and 3.

$$\left|\frac{\pi^2}{6} - S_{10}\right| \approx 0.0951$$

$$\left|\frac{\pi^2}{6} - \left(S_{10} + \frac{21}{220}\right)\right| \approx 0.0002$$

5. Find the number of terms that you would need to ensure an estimate that is accurate to the first 3 decimal places.

$$R_n \leq \frac{1}{n} < \frac{1}{10^3}$$

$$\Rightarrow n > 10^3 = 1000 \text{ terms.}$$

Determine whether the following series converge or diverge.

6. $\sum_{n=1}^{\infty} \frac{2}{5n-1}$

$$\lim_{t\to\infty} \int_1^t \frac{2}{5x-1}\,dx = \lim_{t\to\infty} \frac{2}{5}\ln|5x-1|\Big|_1^t$$

$$= \lim_{t\to\infty} \frac{2}{5}\left(\ln(5t-1) - \ln(4)\right)$$

$$= \infty$$

So $\sum_{n=1}^{\infty} \frac{2}{5n-1}$ $\boxed{\text{diverges by the Integral Test}}$

7. $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$

$u = x^2+1 \Rightarrow \frac{1}{2}du = x\,dx$

$$\lim_{t\to\infty} \int_1^t \frac{x}{x^2+1}\,dx = \lim_{t\to\infty} \int_2^{t^2+1} \frac{1}{2u}\,du$$

$$= \lim_{t\to\infty} \frac{1}{2}\ln|u|\Big|_2^{t^2+1}$$

$$= \lim_{t\to\infty} \frac{1}{2}\left(\ln(t^2+1) - \ln(2)\right)$$

$$= \infty$$

So $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ $\boxed{\text{diverges by the Integral Test}}$

8. $\sum_{n=1}^{\infty} n^2 e^{-n^3}$ $\quad u = -x^3 \Rightarrow \frac{1}{3}du = x^2 dx$

$$\lim_{t\to\infty} \int_1^{-t} \frac{x^2}{e^{x^3}}\,dx = \lim_{t\to\infty} -\frac{1}{3}\int_1^{-t^3} e^u\,du$$

$$= \lim_{t\to\infty} -\frac{1}{3} e^u \Big|_1^{-t^3}$$

$$= \lim_{t\to\infty} -\frac{1}{3}\left(e^{-t^3} - e\right)$$

$$= -\frac{1}{3}(0 - e) = e/3$$

So $\sum_{n=1}^{\infty} n^2 e^{-n^3}$ $\boxed{\text{Converges by the Integral Test}}$

9. $\sum_{n=1}^{\infty} \frac{1}{n^2+4}$

$$\lim_{t\to\infty} \int_1^{t} \frac{1}{x^2+4}\,dx = \lim_{t\to\infty} \frac{1}{2}\arctan\left(\frac{x}{2}\right)\Big|_1^t$$

$$= \lim_{t\to\infty} \frac{1}{2}\left(\arctan\left(\frac{t}{2}\right) - \arctan\left(\frac{1}{2}\right)\right)$$

$$= \frac{1}{2}\left(\pi/2 - \arctan\left(\frac{1}{2}\right)\right) < \infty$$

$\sum_{n=1}^{\infty} \frac{1}{n^2+4}$ $\boxed{\text{converges by the Integral Test}}$