COMPARISON TESTS

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Name: Solutions

Theorem (The Comparison Tests). *Let $\{a_n\}$ and $\{b_n\}$ be sequences, and assume there exists some number N such that*

$$0 < a_n \leq b_n$$

is satisfied whenever $n \geq N$.

(i) If $\sum a_n$ diverges, then $\sum b_n$ also diverges.
(ii) If $\sum b_n$ converges, then $\sum a_n$ also converges.

Theorem (The Limit Comparison Test). *Let $\{a_n\}$ and $\{b_n\}$ be sequences, and assume there exists some number N such that*

$$0 < a_n, b_n$$

is satisfied whenever $n \geq N$. If there exists some number $c > 0$ such that

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$$

then either

- *$\sum a_n$ and $\sum b_n$ both converge, or*
- *$\sum a_n$ and $\sum b_n$ both diverge.*

Decide whether the following series converge or diverge.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{9^n}{3+10^n}$ $10^n \leq 10^n + 3 \Rightarrow \dfrac{1}{3+10^n} \leq \dfrac{1}{10^n} \Rightarrow \dfrac{9^n}{3+10^n} \leq \dfrac{9^n}{10^n}$

So this series converges by comparison with the convergent geometric series

$$\sum_{n=1}^{\infty} \frac{9}{10}\left(\frac{9}{10}\right)^{n-1} = \frac{9}{10\left(1-\frac{9}{10}\right)} = \frac{9}{10-9} = 9$$

2. $\displaystyle\sum_{k=1}^{\infty} \frac{(2k-1)(k^2-1)}{(k+1)(k^2+4)^2}$

$$\frac{(2k-1)(k^2-1)}{(k+1)(k^2+4)^2} = \frac{2k^3-k^2-2k+1}{(k+1)(k^4+8k^2+16)} = \frac{2k^3-k^2-2k+1}{k^5+k^4+8k^3+8k^2+16k+16}$$

So we use the L.C.T. with the convergent p-series with $p=5-3=2$

$$\lim_{k\to\infty} \frac{2k^3-k^2-2k+1}{k^5+k^4+8k^3+8k^2+16k+16} \Big/ \frac{1}{k^2} = \lim_{k\to\infty} \frac{2k^5-k^4-2k^3+k^2}{k^5+k^4+8k^3+8k^2+16k+16}$$

$$=2>0$$

Therefore $\displaystyle\sum_{k=1}^{\infty} \frac{(2k+1)(k^2-1)}{(k+1)(k^2+4)^2}$ converges.

3. $\displaystyle\sum_{n=1}^{\infty} \frac{1+\cos(n)}{e^n}$

Since $-1 \leq \cos(n) \leq 1$ we have

$$\frac{1+\cos(n)}{e^n} \leq \frac{1+1}{e^n} = 2\left(\frac{1}{e}\right)^n = \frac{2}{e}\left(\frac{1}{e}\right)^{n-1}$$

and

$$\sum_{n=1}^{\infty} \frac{2}{e}\left(\frac{1}{e}\right)^{n-1} = \frac{2}{e(1-1/e)} = \frac{2}{e-1}.$$

Therefore $\displaystyle\sum_{n=1}^{\infty} \frac{1+\cos(n)}{e^n}$ converges by Comparison

4. $\displaystyle\sum_{n=1}^{\infty} \frac{2}{\sqrt{n}+2}$

$$\lim_{n\to\infty} \frac{2}{\sqrt{n}+2} \Big/ \left(\frac{1}{\sqrt{n}}\right) = \lim_{n\to\infty} \frac{2\sqrt{n}}{\sqrt{n}+2} = 2$$

So

$\displaystyle\sum_{n=1}^{\infty} \frac{2}{\sqrt{n}+2}$ Diverges by Limit Comparison with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$.

5. $\displaystyle\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$

Since

$$\lim_{x\to\infty} \frac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x}} \overset{L'H}{=} \lim_{x\to\infty} \frac{\cos\left(\frac{1}{x}\right) \frac{d}{dx}\left(\frac{1}{x}\right)}{\frac{d}{dx}\left(\frac{1}{x}\right)}$$

$$= \lim_{x\to\infty} \cos\left(\frac{1}{x}\right)$$

$$= \cos(0)$$

$$= 1$$

we see

$$\lim_{n\to\infty} \frac{\sin(1/n)}{1/n} = 1 > 0$$

So $\displaystyle\sum_{n=1}^{\infty} \sin(1/n)$ diverges by Limit Comparison with the Harmonic Series.