ALTERNATING SERIES

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Name: Solutions

Theorem. *Let $\{b_n\}$ be a sequence with positive terms, $0 < b_n$. If there exists some N such that*

(1) $b_{n+1} \leq b_n$ whenever $n \leq N$ and

(2) $\lim_{n\to\infty} b_n = 0$

then the Alternating Series

$$\sum_{n=1}^{\infty}(-1)^{n-1}b_n$$

converges.

Decide whether the following series converge or diverge.

1. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{3+5n}$ Converges by A.S.T.:

$b_n = \frac{1}{3+5n}$

(1) $\frac{d}{dx}\left(\frac{1}{3+5x}\right) = \frac{-5}{(3+5x)^2} < 0$

$\Rightarrow \frac{1}{3+5(n+1)} < \frac{1}{3+5n}$ for $n \geq 1$.

(2) $\lim_{n\to\infty} \frac{1}{3+5n} = 0$

2. $\sum_{n=1}^{\infty}(-1)^n \frac{3n-1}{2n+1}$ Diverges because $\lim_{n\to\infty} (-1)^n \frac{3n-1}{2n+1}$ Does Not Exist:

$$\lim_{n\to\infty} a_{2n} = \lim_{n\to\infty} \frac{3(2n)-1}{2(2n)-1} = \frac{3}{2}$$

$$\lim_{n\to\infty} a_{2n+1} = \lim_{n\to\infty} \frac{-3(2n+1)-1}{2(2n+1)-1} = -3/2$$

3. $\displaystyle\sum_{n=1}^{\infty}(-1)^n \sin\left(\frac{\pi}{n}\right)$ Converges by A.S.T.:

$b_n = \sin(\pi/n)$

① $\frac{d}{dx}\sin(x) = \cos(x) > 0$

on $(0, \pi/2)$, so when $n \geq 3$

$0 < \frac{\pi}{n+1} < \frac{\pi}{n} < \frac{\pi}{2}$

$\Rightarrow b_{n+1} = \sin\left(\frac{\pi}{n+1}\right) < \sin\left(\frac{\pi}{n}\right) = b_n$

② $\lim_{n\to\infty} \sin\left(\frac{\pi}{n}\right) = \sin\left(\lim_{n\to\infty}\frac{\pi}{n}\right)$

$= \sin(0)$

$= 0.$

4. $\displaystyle\sum_{n=1}^{\infty}(-1)^n \frac{n^2}{n^2+n+1}$ Diverges because $\lim_{n\to\infty} a_n$ Does Not Exist:

$$\lim_{n\to\infty} a_{2n} = \lim_{n\to\infty} \frac{(2n)^2}{(2n)^2+(2n)+1} = 1$$

$$\lim_{n\to\infty} a_{2n+1} = \lim_{n\to\infty} -\frac{(2n+1)^2}{(2n+1)^2+(2n+1)+1} = -1.$$

5. For what values of p is the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^p}$$

convergent?

$b_n = \frac{1}{n^p}$

when $p > 0$

$\frac{d}{dx}\left(\frac{1}{x^p}\right) = \frac{-p}{x^{p+1}} < 0 \Rightarrow b_{n+1} \leq b_n$ and

$\lim_{n\to\infty} \frac{1}{n^p} = 0$

so

$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^p}$ converges when $p > 0$

6. Approximate the sum of the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^6}$$

correct to four decimal places.

$|R_n| \leq \frac{1}{(n+1)^6} < \frac{1}{10^4} \iff 10^4 < (n+1)^6 \iff 10^{2/3} \approx 4.6 < n+1$

If we take $n \geq 4$, then we know $|R_n| \leq \frac{1}{5^6} < \frac{1}{10^4}$.

$S_4 = 1 - \frac{1}{2^6} + \frac{1}{3^6} - \frac{1}{4^6} \approx 0.9855.$