RELATED RATES

BLAKE FARMAN

Lafayette College

Name: Solutions

1. Gas is escaping a spherical balloon at the rate of 4 cm^3 per minute. How fast is the surface area shrinking when the radius is 24 cm? For a sphere, $V = \frac{4}{3}\pi r^3$ and $S = 4\pi r^2$ where V is volume, S is surface area and r is the radius of the balloon.

$$\frac{dS}{dt} = 4\pi \frac{d}{dt}(r^2) = 4\pi\left(2r\frac{dr}{dt}\right) = 8\pi r\frac{dr}{dt},$$

so to find $\frac{dS}{dt}$ we need to know $\frac{dr}{dt}$. Compute

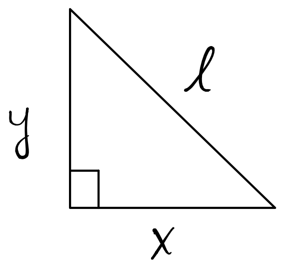
$$\frac{dV}{dt} = \frac{4}{3}\pi\frac{d}{dt}r^3 = \frac{4}{3}\pi\, 3r^2\frac{dr}{dt} = 4\pi r^2\frac{dr}{dt}$$

$$\Rightarrow \frac{dr}{dt} = \frac{1}{4\pi r^2}\frac{dV}{dt}$$

$$\Rightarrow \frac{dS}{dt} = 8\pi r\left(\frac{1}{4\pi r^2}\right)\frac{dV}{dt} = \frac{2}{r}\frac{dV}{dt}$$

$$= \frac{2}{24}(-4) = \frac{-2}{6} = \boxed{-\frac{1}{3}\ cm^2/s}$$

2. The top of a ladder slides down a vertical wall at a rate of 0.15 meters/second. At the moment when the bottom of the ladder is 3 meters from the wall, it slides away from the wall at a rate of 0.2 meters/second. How long is the ladder?



Given: $\frac{dy}{dt} = \frac{-15}{100}\ \text{m/s} = \frac{-3}{20}\ \text{m/s}$ $\quad x=3$

$\frac{dx}{dt} = \frac{2}{10}\ \frac{m}{s} = \frac{1}{5}\ \text{m/s}$

Know $\frac{d\ell}{dt}=0$ because the ladder is a constant length.

$$x^2+y^2=\ell^2 \Rightarrow 2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2\ell\frac{d\ell}{dt} = 0$$

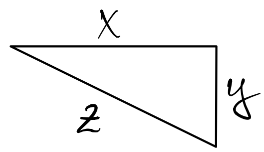
$$\Rightarrow y = \frac{-2x\frac{dx}{dt}}{2\frac{dy}{dt}} = \frac{-x\frac{dx}{dt}}{\frac{dy}{dt}}$$

$$= \frac{-3\left(\frac{1}{5}\right)}{\left(\frac{-3}{20}\right)} = \frac{3}{5}\left(\frac{20}{3}\right) = \frac{20}{5} = 4$$

So

$$\ell = \sqrt{x^2+y^2} = \sqrt{9+16} = \sqrt{25} = \boxed{5m}$$

3. Two cars start moving from the same pont. One travels south at 60 mi/h and the other travels west at 25 mi/h. At what rate is the distance between the cars increasing two hours later?



Given: $y' = \frac{dy}{dt} = 60 \text{mi/h}$, $x' = \frac{dx}{dt} = 25 \text{mi/h}$

Want $z' = \frac{dz}{dt}$ when $t = 2$.

$$x = 2x', \quad y = 2y', \quad z = \sqrt{4(x')^2 + 4(y')^2} = 2\sqrt{(x')^2 + (y')^2}$$

$$x^2 + y^2 = z^2 \Rightarrow 2xx' + 2yy' = 2zz' \Rightarrow xx' + yy' = zz'$$

$$\Rightarrow z' = \frac{xx' + yy'}{z} = \frac{2(x')^2 + 2(y')^2}{2\sqrt{(x')^2 + (y')^2}} = \frac{\left((x')^2 + (y')^2\right)^1}{\left((x')^2 + (y')^2\right)^{1/2}}$$

$$= \left((x')^2 + (y')^2\right)^{1 - 1/2} = 2\sqrt{(x')^2 + (y')^2}$$

$$= \sqrt{(25)^2 + (60)^2} = 2\sqrt{5^4 + 2^4 3^2 5^2}$$

$$= \sqrt{5^2(25 + 144)}$$

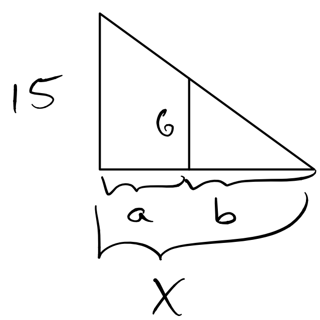
$$= \sqrt{5^2}\sqrt{169}$$

$$= (5)\sqrt{13^2}$$

$$= 5(13)$$

$$= \boxed{65 \text{ mph}}$$

4. A street light is mounted at the top of a 15-ft-tall pole. A man 6 ft tall walks away from the pole with a speed of 5 ft/s along a straight path. How fast is the tip of his shadow moving when he is 40 ft from the pole?
(Hint: The length of the shadow is measured from the person to the tip of the shadow; the rate at which the tip of the shadow is moving is measured from the pole to the tip of the shadow.)



Given: $a' = \frac{da}{dt} = 5\text{ft/s}$

Want $X' = \frac{dx}{dt}$

$$\frac{b}{6} = \frac{x}{15}, \quad b = x - a$$

$$6x = 15b = 15(x-a) = 15x - 15a$$

$$\Rightarrow -9x = 15a$$

$$\Rightarrow x = \frac{-15}{-9}a = \frac{5}{3}a$$

$$\Rightarrow x' = \frac{5}{3}a' = \frac{5}{3}(5) = \boxed{\frac{25}{3}}$$