**OPTIMIZATION**

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Name: Solutions

## Closed Inteveral Method

Find the absolute maximum and minimum values of the function on the specified interval.

**1.** $f(x) = 5 + 54x - 2x^3$, $[0, 4]$

$f'(x) = 54 - 6x^2 = 6(9 - x^2) = 6(x-3)(x+3) = 0 \iff x = 3$ or $x = -3$.

$f(x) = 5 + 2x(27 - x^2)$

$f(0) = \boxed{5}$ min

$f(3) = 5 + 6(27-9) = 5 + 6(18) = 5 + 60 + 48 = \boxed{113}$ max

$f(4) = 5 + 8(27-16) = 5 + 8(11) = 5 + 88 = 93$

**2.** $f(x) = x^3 - 6x^2 + 5$, $[-3, 5]$

$f'(x) = 3x^2 - 12x = 3x(x-4) = 0 \iff x = 0$ or $x = 4$

$f(x) = x^3 - 6x^2 + 5 = x^2(x-6) + 5$

$f(-3) = 9(-9) + 5 = -81 + 5 = \boxed{-76}$ min

$f(0) = \boxed{5}$ max

$f(4) = 16(-2) + 5 = -32 + 5 = -27$



$f(5) = 25(-1) + 5 = -20$

**3.** $f(t) = (t^2-4)^3$,$[-2,3]$

$f'(t) = 3(t^2-4)^2(2t) = 6t(t^2-4)^2 = 0 \iff t = 0, 2, \text{or } -2$

$t^2-4 = (t+2)(t-2) \Rightarrow f(2) = f(-2) = 0$

$f(0) = (-4)^3 = \boxed{-64}$ min.

$f(3) = (9-4)^3 = 5^3 = \boxed{125}$ max

**4.** $f(x) = 2x^3 - 3x^2 - 12x + 1$, $[-2,3]$

$f'(x) = 6x^2 - 6x - 12 = 6(x^2 - x - 2) = 6(x-2)(x+1) = 0 \iff x = 2 \text{ or } x = -1.$

$f(x) = 2x^3 - 3x^2 - 12x + 1 = x(2x^2 - 3x - 12) + 1$

$f(-2) = -2(8+6-12)+1 = -2(2)+1 = -4+1 = -3$

$f(-1) = -(2+3-12)+1 = -(-7)+1 = \boxed{8}$ max.

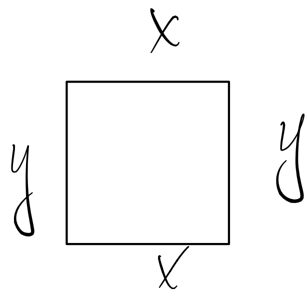
$f(2) = 2(8-6-12)+1 = 2(2-12)+1 = -20+1 = \boxed{-19}$ min

$f(3) = 3(18-9-12)+1 = 3(9-12)+1 = 3(-3)+1 = -8$

## Optimization

**5.** A farmer has 2400 feet of fencing and wants to use it to fence off a rectangular field. What are the dimensions of the field that has the largest area, and what is that largest area?

Step 1. Draw a picture of a possible field. Label the picture by assigning variables to any quantities that change. List any other variables that might be important.




$A$ = area

$P$ = perimeter

Step 2. Which quantity from the previous part is the one that we want to maximize?

$A$

Step 3. Use your picture from Step 1 to write a formula for the variable that you named in Step 2. How many independent variables does this function have?

$A = xy$; 2 independent variables.

Step 4. Model the constraint that we have only 2400 feet of fencing using an equation involving the variables from Step 1. Use this equation to rewrite the equation in Step 3 as a function of a *single* variable.

$P = 2x + 2y = 2400 \Rightarrow x = \frac{2400 - 2y}{2} = 1200 - y$

$A = (1200 - y)y = 1200y - y^2$

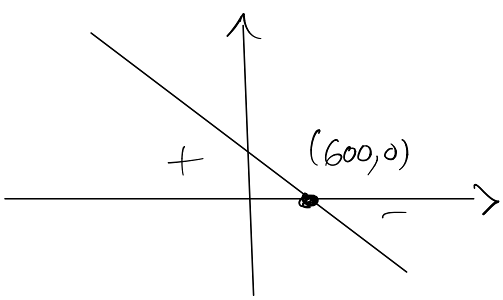
Step 5. What is the domain of this function? [Hint: The following steps will be easier if you allow degenerate rectangles with zero area.]

$[0, 1200]$

Step 6. Use the Closed Interval Method to find the absolute maximum value on the domain from Step 5.

$A' = 1200 - 2y = 0 \Leftrightarrow 2y = 1200$

$\Leftrightarrow y = 600$




so this is a local max.

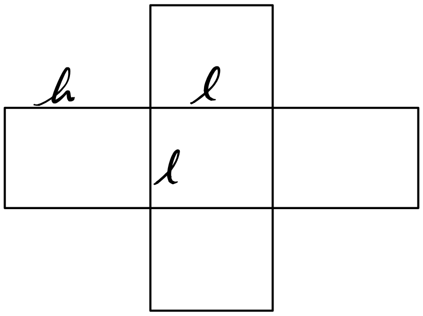
Step 7. What are the dimensions of the field that has the largest area, and what is the largest area?

$x = 600, \quad y = 600$

$A = 360{,}000.$

**6.** A box with a square base and no top must have a volume of $32\text{cm}^3$. Find the dimensions of the box that minimize the amount of material used.

Step 1. Imagine that you flatten a box with no top by cutting the edges between each of the sides. Draw this flattened box and assign variables to any quantities that change.




V = volume

S = surface area.

Step 2. The surface area of the box is the sum of the areas of all its sides. Use your picture to write down an equation for the surface area, $S$, of the box.

$$S = \ell^2 + 4\ell h$$

Step 3. Recall that the volume of a cube with length $\ell$, width $w$, and height $h$ is $V = \ell \cdot w \cdot h$. Use this and your picture to write down an equation for the volume, $V$, of the box.

$$V = \ell^2 h = 32$$

Step 4. Use your answers to Step 2 and Step 3 to express the surface area as a function of a *single* variable.

$$h = \frac{32}{\ell^2} \Rightarrow S = \ell^2 + 4\ell\left(\frac{32}{\ell^2}\right) = \ell^2 + \frac{128}{\ell} = \ell^2 + 128\ell^{-1}$$

Step 5. Use the First Derivative Test for Absolute Extreme Values to find the dimensions corresponding to a minimum surface area.

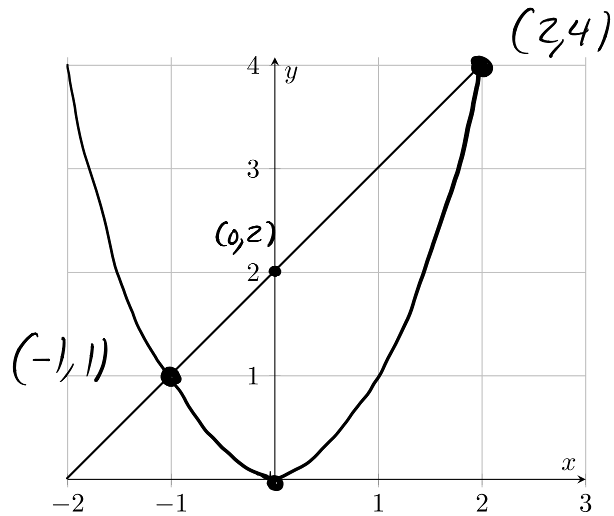
$$0 = S' = 2\ell - 128/\ell^2 \Leftrightarrow 2\ell^3 - 128 = 0 \Leftrightarrow \ell^3 = 128/2 = 64 = 4^3$$

$$\Leftrightarrow \ell = 4, \; h = \frac{32}{16} = 2.$$

$S'' = 2 + 256\ell^{-3} > 0$ when $\ell > 0$, so $\ell = 4$ is a minimum.

**7.** What is the maximum vertical distance between the line $y = x+2$ and the parabola $y = x^2$ for $-1 \le x \le 2$?

Step 1. Sketch a graph of each function in the plane.




Step 2. Plot the point on the line and the parabola when $x = 0$. What is the vertical distance between these two points?

$2 - 0 = 2.$

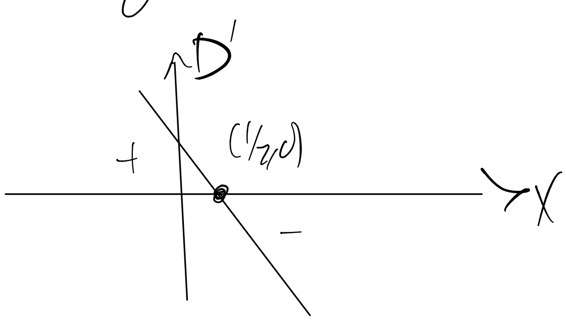
Step 3. Write down a function that takes in an $x$-value, and outputs the vertical distance between the line and the parabola.
**Note:** This function should output your answer from Step 2 when you evaluate it at 0.

$D = (x+2) - x^2$

Step 4. Identify the domain of this function, then use the Closed Interval Method to find the maximum vertical distance between the line and the parabola.

$[-1, 2]$

$0 = D' = 1 - 2x \iff x = 1/2$

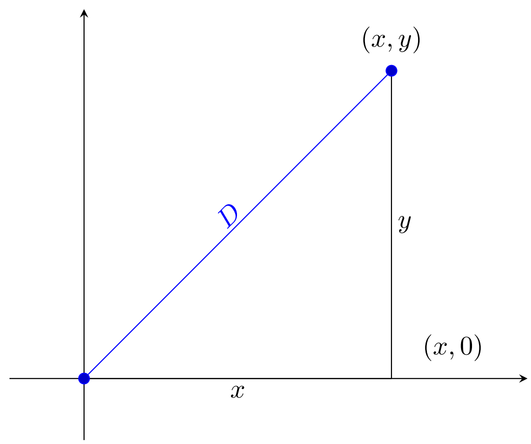



$D(1/2) = \frac{1}{2} + 2 - \frac{1}{4}$

$= \frac{1}{4} + \frac{8}{4} = \boxed{\frac{9}{4}}$

**8.** Find the point on the line $y = 2x + 3$ that is closest to the origin.

Step 1. Recall that the distance between the points $(x, y)$ and $(0, 0)$ is the length of the line segment connecting them:




Write down an equation expressing the distance, $D$, between the point $(x, y)$, and the point $(0, 0)$.

$D^2 = x^2 + y^2 \Rightarrow D = \sqrt{x^2 + y^2}$

Step 2. Use the line $y = 2x + 3$ to transform your answer to Step 1 into an equation involving only the variable $x$.

$D = \sqrt{x^2 + (2x+3)^2}$

Step 3. Explain in words what the quantity $D$ from Step 2 represents, and how the function $D(x)$ is relevant to this problem.

This is a function that measures the distance between the origin and a point on the line, which is exactly the function we need to minimize.

Step 4. Square both sides of the equation from Step 2, then use Implicit Differentiation to find $D'(x)$.

*Remark:* While you could compute the derivative directly, this makes things simpler by removing square roots!

$$D^2 = x^2 + (2x+3)^2$$

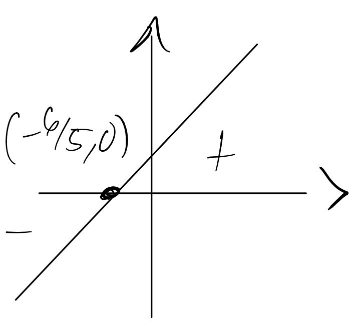
$$\Rightarrow 2DD' = 2x + 2(2x+3)(2) = 2x + 4(2x+3)$$

$$= 2x + 8x + 12 = 2(5x+6)$$

$$\Rightarrow D' = \frac{5x+6}{D} = \frac{5x+6}{\sqrt{x^2+(2x+3)^2}}$$

Step 5. Use your answers to the previous parts to find the point on $y = 2x + 3$ that is closest to the origin.

$$0 = D' \Leftrightarrow 5x+6 = 0 \Leftrightarrow x = -6/5$$




$$y = 2(-6/5) + 3 = \frac{-12}{5} + \frac{15}{5} = \frac{3}{5}$$

$$\boxed{(-6/5, 3/5)}$$