MEAN VALUE THEOREM

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Name: Solutions

Theorem (Mean Value)**.** *Let f be a function that satisfies the following hypotheses:*

(1) f is continuous on the closed interval $[a, b]$.

(2) f is differentiable on the open interval (a, b).

Then there is a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

or, equivalently,

$$f(b) - f(a) = f'(c)(b - a).$$

In Problems 1 through 4, verify that the function satisfies the hypotheses of the Mean Value Theorem on the given interval and find all numbers c that satisfy its conclusion.

1. $f(x) = x^3 - x^2 - 6x + 2$, $[0, 3]$

$f(0) = 2 \quad f(3) = 3^3 - 3^2 - 6(3) + 2 = 2$

By the M.V.T., there is some $0 < c < 3$ satisfying

$$f'(c) = 3c^2 - 2c - 6 = \frac{2-2}{3-0} = 0$$

$$c = \frac{2 \pm \sqrt{4 + 4(3)(6)}}{6} = \frac{2 \pm 2\sqrt{1+18}}{6} = \frac{1 \pm \sqrt{19}}{3}$$

Since $\sqrt{16} = 4 < \sqrt{19} < \sqrt{25} = 5$ we have

$$\frac{1+4}{3} = \frac{5}{3} < \frac{1+\sqrt{19}}{3} < \frac{1+5}{3} = 2 \quad \text{and} \quad \frac{1-5}{3} = \frac{-4}{3} < \frac{1-\sqrt{19}}{3} < \frac{1-4}{3} = -1$$

So $\boxed{c = \frac{1+\sqrt{19}}{3}}$ is the only solution to $f'(c) = 0$ on $(0, 3)$.

2. $f(x) = \cos(2x)$, $[\pi/8, 7\pi/8]$

$f(\pi/8) = \cos(\pi/4) = \frac{\sqrt{2}}{2}$, $f(7\pi/4) = \cos(7\pi/4) = \frac{\sqrt{2}}{2}$

So by the M.V.T. there is some $\pi/8 < c < 7\pi/8$ such that

$$f'(c) = -2\sin(2x) = \frac{\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}}{\frac{7\pi}{8} - \pi/8} = 0$$

Since $\sin(x) = 0$ whenever $x = n\pi$, n an integer, the only solution on $(\pi/8, 7\pi/8)$ is $\boxed{c = \pi/2}$.

$$-2\sin\left(2\left(\frac{\pi}{2}\right)\right) = -2\sin(\pi) = -2(0) = 0.$$

3. $f(x) = 3x^2 + 2x + 5$, $[-1, 1]$

$f(1) = 3 + 2 + 5 = 10$, $f(-1) = 3 - 2 + 5 = 6$

MVT $\Rightarrow$ $f'(c) = 6c + 2 = \frac{10-6}{1-(-1)} = \frac{4}{2} = 2$

So $\boxed{c = 0}$

4. $f(x) = \dfrac{x}{x+2}$, $[1, 4]$

$f(1) = \frac{1}{1+2} = \frac{1}{3}$, $f(4) = \frac{4}{4+2} = \frac{4}{6} = \frac{2}{3}$, $f'(x) = \frac{(x+2) - x}{(x+2)^2} = \frac{2}{(x+2)^2}$

$$f'(c) = \frac{2}{(c+2)^2} = \frac{\frac{2}{3} - \frac{1}{3}}{4 - 1} = \frac{\frac{1}{3}}{3} = \frac{1}{9}$$

$\Rightarrow 18 = (c+2)^2 \Rightarrow c + 2 = \pm\sqrt{18} = \pm\sqrt{2}\sqrt{9} = \pm 3\sqrt{2}$

$\Rightarrow c = -2 \pm 3\sqrt{2}$

Since $-2 - 3\sqrt{2} < 0$, the only solution on $(1, 4)$ must be

$$\boxed{c = -2 + 3\sqrt{2}.}$$

Note: Even though we don't need to, we can check this explicitly. $1 < \sqrt{2} < 2 \Rightarrow 3 < 3\sqrt{2} < 6 \Rightarrow -2+3 = 1 < -2 + 3\sqrt{2} < -2 + 6 = 4.$