CONTINUITY

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Name: Solutions

Definition. A function, f, is **continuous** at a if

$$\lim_{x\to a} f(x) = f(a)$$

1. Use the definition to show that the given function is continuous at the given number, a.

(a) $f(t) = \dfrac{t^2+5t}{2t+1}$, $a = 2$.

$$f(2) = \frac{2^2+5(2)}{2(2)+1} = \frac{4+10}{4+1} = \frac{14}{5}$$

$$\lim_{t\to 2} f(t) = \lim_{t\to 2} \frac{t^2+5t}{2t+1} = \frac{2^2+5(2)}{2(2)+1} = \frac{14}{5} = f(2).$$

(b) $f(x) = 3x^4 - 5x + \sqrt[3]{x^2+4}$, $a = 2$.

$$f(2) = 3(2)^4 - 5(2) + \sqrt[3]{2^2+4} = 3(16) - 10 + \sqrt[3]{8} = 38 + 2 = 40$$

$$\lim_{x\to 2} f(x) = 3(2)^4 - 5(2) + \sqrt[3]{2^2+4} = 3(16) - 10 + \sqrt[3]{8} = 40 = f(2).$$

2. Show that the function

$$f(x) = \frac{x-1}{3x+6}$$

is continuous on the interval $(-\infty, -2) \cup (-2, \infty)$.

When $3x+6 \neq 0$, the Limit Laws tell us

$$\lim_{x \to a} f(x) = \lim_{x \to a} \frac{x-1}{3x+6} = \frac{\lim_{x \to a} x-1}{\lim_{x \to a} 3x+6} = \frac{a-1}{3a+6} = f(a)$$

So f is continuous except when

$$3x+6=0 \Rightarrow 3x=-6 \Rightarrow x=-6/3=-2.$$

3. Find the number k that makes the function

$$f(x) = \begin{cases} \dfrac{x^3-8}{x^2-4} & x \neq 2 \\ k & x = 2 \end{cases}$$

continuous.

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} \frac{x^3-8}{x^2-4} = \lim_{x \to 2} \frac{(x-2)(x^2+2x+4)}{(x-2)(x+2)}$$

$$= \lim_{x \to 2} \frac{x^2+2x+4}{x+2} = \frac{4+4+4}{4} = \frac{12}{4} = 3$$

So

$$k = f(2) = \lim_{x \to 2} f(x) = \boxed{3}$$

4. Use continuity to evaluate the given limit.

(a) $\lim_{x\to\pi} \sin\big(x + \sin(x)\big)$

$$= \sin\left(\lim_{x\to\pi} x + \sin(x)\right)$$

$$= \sin(\pi + \sin(\pi))$$

$$= \sin(\pi + 0)$$

$$= \sin(\pi)$$

$$= \boxed{0}$$

(b) $\lim_{x\to 4} x\sqrt{20 - x^2}$ $= \lim_{x\to 4} x \sqrt{\lim_{x\to 4}(20 - x^2)}$

$$= 4\sqrt{20 - 4^2}$$

$$= 4\sqrt{20-16}$$

$$= 4\sqrt{4}$$

$$= 4(2)$$

$$= \boxed{8}$$

5. Use the Intermediate Value Theorem to show that there is a solution to the given equation in the specified interval.

Note: You do not need to find the solution!

(a) $x^4 + x - 3 = 0$, $(1, 2)$

$f(x) = x^4 + x - 3$ is continuous, $f(1) = 1 + 1 - 3 = 2 - 3 = -1 < 0$, and

$f(2) = 2^4 + 2 - 3 = 16 - 1 = 15 > 0$, so $f(c) = 0$ for some $1 < c < 2$.

(b) $\frac{2}{x} = x - \sqrt{x}$, $(2, 3)$

$f(x) = \frac{2}{x} - x + \sqrt{x}$ is continuous on $(2, 3)$, $f(2) = 1 - 2 + \sqrt{2} = \sqrt{2} - 1 > 0$,

and $f(3) = \frac{3}{2} - 3 + \sqrt{3} = \frac{-3}{2} + \sqrt{3} < \frac{-3}{2} + 2 = \frac{-1}{2} < 0$, so

$f(c) = 0$ for some $2 < c < 3$.

(c) $\cos(x) = x$, $(0, 1)$

$f(x) = \cos(x) - x$ is continuous, $f(0) = \cos(0) - 0 = 1$, and

$f(1) = \cos(1) - 1 < \cos(0) - 1 = 1 - 1 = 0$, so

$f(c) = 0$ for some $0 < c < 1$.