4/15/16 ①

$$2x + 3y = 4$$
$$5x + 7y = 4$$

a) $A = \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$

b) what is the inverse of $\begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix}$?

$$\frac{1}{14-15} \begin{bmatrix} 7 & -3 \\ -5 & 2 \end{bmatrix} = -1 \begin{bmatrix} 7 & -3 \\ -5 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -7 & 3 \\ 5 & -2 \end{bmatrix} = B$$

For a 2x2 matrix, $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the inverse is

$$\frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

c) Solve the system.

$$\begin{bmatrix} -7 & 3 \\ 5 & -2 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -7 & 3 \\ 5 & -2 \end{bmatrix} \begin{bmatrix} 4 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -7 & 3 \\ 5 & -2 \end{bmatrix}\begin{bmatrix} 4 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -7 & 3 \\ 5 & -2 \end{bmatrix}\begin{bmatrix} 4 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} -16 \\ 12 \end{bmatrix}.$$

So ~~my~~ the solution is $(-16, 12)$.

$$6\left(\frac{x}{2} + \frac{y}{3}\right) = (-5)6 \iff 3x + 2y = -30$$

$$3\left(\frac{x}{3} + y\right) = (-8)3 \qquad x + 3y = -24$$

$$\begin{bmatrix} 3 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -30 \\ -\cancel{3}24 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 1 & 3 \end{bmatrix}^{-1} \cdot \begin{bmatrix} -30 \\ -24 \end{bmatrix}.$$

In general, for any real numbers a, b, c, d, e, f such that $ad - bc \neq 0$

The solution to the system

$$\begin{aligned} ax + by &= e \\ cx + dy &= f \end{aligned} \iff \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix}$$

is given by

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} e \\ f \end{bmatrix}$$

Analogue: If a is any real number,

$$a \cdot a^{-1} = a \cdot \left(\frac{1}{a}\right) = \frac{a}{a} = 1.$$

2) $$\begin{aligned} 2x + 3y &= 9 \\ 5x + 7y &= 19. \end{aligned}$$

$$\begin{bmatrix} -7 & 3 \\ 5 & -2 \end{bmatrix}\begin{bmatrix} 9 \\ 19 \end{bmatrix} = \begin{bmatrix} -6 \\ 7 \end{bmatrix}$$

3) $$\begin{aligned} x - y + 7z &= 4 \\ x - y + 8z &= 3 \end{aligned} \iff \left[\begin{array}{ccc|c} 1 & -1 & 7 & 4 \\ 1 & -1 & 8 & 3 \end{array}\right]$$

reduced form $\left[\begin{array}{ccc|c} 1 & -1 & 0 & 11 \\ 0 & 0 & 1 & -1 \end{array}\right] \iff \begin{aligned} x - y &= 11 \\ z &= -1 \end{aligned}$

$x - y = 11 \Rightarrow x - 11 = y$

Solutions: $\{(x, x-11, -1) \mid x \in \mathbb{R}\}$

By hand:

$$\left[\begin{array}{ccc|c} 1 & -1 & 7 & 4 \\ 1 & -1 & 8 & 3 \end{array}\right] \xrightarrow{R_2 - R_1} \left[\begin{array}{ccc|c} 1 & -1 & 7 & 4 \\ 0 & 0 & 1 & -1 = 3-4 \end{array}\right]$$

$$\xrightarrow{R_1 - 7R_2} \left[\begin{array}{ccc|c} 1 & -1 & 0 & 4-7(-1) \\ 0 & 0 & 1 & -1 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & -1 & 0 & 11 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

4) $2x - y = 0$

$x + y + z = 18$

$x - z = 2$

$$\longleftrightarrow \left[\begin{array}{ccc|c} 2 & -1 & 0 & 0 \\ 1 & 1 & 1 & 18 \\ 1 & 0 & -1 & 2 \end{array}\right]$$

$$\xrightarrow[2R_3 - R_1]{2R_2 - R_1} \left[\begin{array}{ccc|c} 2 & -1 & 0 & 0 \\ 0 & 3 & 2 & 36 \\ 0 & 1 & -2 & 4 \end{array}\right] \xrightarrow[3R_3 - R_2]{3R_1 + R_2} \left[\begin{array}{ccc|c} 6 & 0 & 2 & 36 \\ 0 & 3 & 2 & 36 \\ 0 & 0 & -8 & -24 \end{array}\right]$$

$$\xrightarrow{\frac{-1}{8}R_3}\left[\begin{array}{ccc|c} 6 & 0 & 2 & 36 \\ 0 & 3 & 2 & 36 \\ 0 & 0 & 1 & 3 \end{array}\right] \xrightarrow[R_2-2R_3]{R_1-2R_3} \left[\begin{array}{ccc|c} 6 & 0 & 0 & 30 \\ 0 & 3 & 0 & 30 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$\xrightarrow[\frac{1}{3}R_2]{\frac{1}{6}R_1}\left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 10 \\ 0 & 0 & 1 & 3 \end{array}\right] \qquad (5, 10, 3).$$

$$2x - y = 0$$

$$2(x + y + z) = 18 \cdot 2$$

$$\begin{array}{r} 2x + 2y + 2z = 36 \\ - \quad 2x - y + 0 \cdot z = 0 \\ \hline 0 \quad 3y + 2z = 36 \end{array}$$

$$\begin{bmatrix} 2 & -1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 18 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \underbrace{\begin{bmatrix} 2 & -1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix}^{-1}}_{A^{-1}} \underbrace{\begin{bmatrix} 0 \\ 18 \\ 2 \end{bmatrix}}_{B} = \begin{bmatrix} 5 \\ 10 \\ 3 \end{bmatrix}$$

3: $\left[\begin{array}{ccc|c} 1 & 3 & 1 & 4 \\ 3 & 0 & 3 & 12 \end{array}\right] \xrightarrow{RREF} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 0 & 1 & 0 & 0 \end{array}\right]$

$x + z = 4$ $\quad 1\cdot x + 0\cdot y + 1\cdot z = 4$

$y = 0$ $\quad 0\cdot x + 1\cdot y + 0\cdot z = 0.$

$z = 4 - x.$

$\{(x, 0, 4-x) \mid x \in \mathbb{R}\}$

E.g.: $x = 5$

$(5, 0, -1)$

4: $x + \cancel{y}\, z = 4$

$2y = 4$

$3x - 3z = -6$

$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 0 & 2 & 0 & 4 \\ 3 & 0 & -3 & -6 \end{array}\right]$

$\xrightarrow{R_3 - 3R_1} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 0 & 2 & 0 & 4 \\ 0 & 0 & -6 & -18 \end{array}\right] \xrightarrow[\frac{1}{2}R_2]{-\frac{1}{6}R_3} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$

$$\xrightarrow{R_1 - R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right] \qquad (1, 2, 3).$$

Logic

Truth Tables

Negation

p	$\neg p$
T	F
F	T

Disjunction (Or)

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Conjunction (And)

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Implication

p	q	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Contrapositive

The contrapositive of the implication

$$p \Rightarrow q$$

is

$$\neg q \Rightarrow \neg p.$$

These are logically equivalent:

$$p \Rightarrow q \equiv \neg q \Rightarrow \neg p.$$

Show this with a truth table:

p	q	$p \Rightarrow q$	$\neg q$	$\neg p$	$\neg q \Rightarrow \neg p$
T	T	T	F	F	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

Same columns means same truth value no matter the value of p & q.

Write the truth table for (4)

$$(p \wedge q) \Rightarrow (p \vee q)$$

p	q	$(p \wedge q)$	$p \vee q$	$(p \wedge q) \Rightarrow (p \vee q)$
T	T	T	T	T
T	F	F	T	T
F	T	F	T	T
F	F	F	F	T

Tautology.

p	q	$p \vee q$	$p \wedge q$	$(p \vee q) \Rightarrow (p \wedge q)$
T	T	T	T	T
T	F	T	F	F
F	T	T	F	F
F	F	F	F	T

4/22/16 (1)

3 red marbles
2 green marbles
1 lavender marbles
2 yellow marbles
2 orange marbles.

10 marbles total

How many sets of four?

$$\binom{10}{4} = \frac{10!}{(10-4)!\,4!} = \frac{10!}{6!\,4!} = \frac{10\cdot 9\cdot \cancel{8}\cdot 7}{\cancel{4}\cdot 3\cdot \cancel{2}} = 10\cdot 3\cdot 7$$
$$= 210.$$

$$P(n,r) = \frac{n!}{(n-r)!}$$

Order matters

$$\binom{n}{r} = C(n,r) = \frac{n!}{(n-r)!\,r!}$$

Order doesn't matter.

How many sets of four include all the red ones?

1. Take all red marbles - $\binom{3}{3} = 1$ way to do this
2. ~~3~~ Choose a fourth marble that's not red. There are $\binom{7}{1} = \frac{7!}{(7-1)!\,1!} = \frac{7\cdot\cancel{6!}}{\cancel{6!}} = 7$ ways to do this

#2

There are 7 ways to choose a set of four with all the red ones.

How many sets of four include at most one red marble?

Either we have no red marbles <u>or</u> we have 1 red marble.

<u>Case 1</u>: no red marbles

$$\binom{7}{4} = \frac{7!}{(7-4)!\,4!} = \frac{7!}{3!\,4!} = \frac{7\cdot\cancel{6}\cdot 5}{\cancel{3}\cdot\cancel{2}} = 35$$

ways to choose a set of four with no red marbles.

<u>Case 2</u>: one red marble.

1. Choose a red marble.

$$\binom{3}{1} = \frac{3!}{(3-1)!\,1!} = \frac{3!}{2} = 3$$

ways to do this

2. From the remaining 7 marbles that are not red, choose 3.

$$\binom{7}{3} = \frac{7!}{(7-3)!\,3!} = \frac{7!}{4!\,3!} = 35$$

ways to do this.

So there are

$$35 \cdot 3 = 105$$

ways to choose a set of four with one red

There are

$$35 + 105 = 140$$

ways to choose a set of four marbles with at most one red.

How many three letter sequences are there that use the letters

$$q, u, a, k, e, s$$

at most once each?

Have 6 letters, want to know how many 3-permutations of the 6 letters:

$$P(6,3) = \frac{6!}{(6-3)!} = \frac{6!}{3!} = 6 \cdot 5 \cdot 4 = 120.$$

How many six letter sequences are possible that use the letters

$$q, u, a, a, k, u?$$

Think of it this way:

- Find how many ways there are to "place" the a's
- Find how many ways there are to "place" the u's for each choice of location of the a's
- The location of the k is determined.

_ _ _ _ _ _

$$\binom{6}{3} = \frac{6!}{(6-3)!\,3!} = \frac{6!}{3!\,3!} = \frac{6\cdot 5\cdot 4}{6} = 20$$

ways to choose a location for the a's.

For each choice of location for the a's there are $\binom{3}{2} = \frac{3!}{1!\,2!} = 3$ ways to choose a location for the u's, and then only one location for the K.

So there are 60 such sequences.

$S = \{\text{all cars}\}$

$T = \{\text{all blue cars}\} \subseteq S$

$U = \{\text{all toyotas}\} \subseteq S$

Explain in words the set

$T \cup U$

This is the set of all cars that are either blue or a toyota. (or both)

$T \cap U$

This is the set of all blue toyotas

① $S \setminus T \cup S \setminus U$ ② $S \setminus T \cap S \setminus U$

$S \setminus T$ - set of all cars that are not blue

$S \setminus U$ - set of all cars that are not toyotas

① is the set of all cars that are either not blue or not toyotas.

② is the set of all cars that are not blue and not toyotas (e.g. green Ford).