Solve the 2x2 game

$$P = \begin{bmatrix} -2 & 0 \\ -3 & 1 \end{bmatrix}$$

Two players A, B - A the row player, B the column player. The strategy for A is

$$R = [x \quad 1-x]$$

the strategy for B is

$$C = \begin{bmatrix} y \\ 1-y \end{bmatrix}$$

The expected payoff is

$$e(x,y) = [x \quad 1-x] \begin{bmatrix} -2 & 0 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} y \\ 1-y \end{bmatrix}$$

$$= [x \quad 1-x] \begin{bmatrix} -2y \\ -3y + (1-y) \end{bmatrix}$$

$$= -2xy + (1-x)(-4y+1)$$

$$= -2xy + 4y + 1 + 4xy - x$$

$$= 2xy - 4y - x + 1.$$

First, we'll find the optimal strategy for player A.

If we choose a strategy for player A, $x = x_0$
e.g. $x = 1/3$, then

$$e(y) = e(x_0, y) = 2x_0 y - 4y - x_0 + 1 = (2x_0 - 4)y - (x_0 + 1)$$

$x_0 = 1/3$

$$e(y) = (2(\tfrac{1}{3}) - 4)y - (\tfrac{1}{3} + 1) = -\tfrac{10}{3}y - 4/3.$$

This expected payoff function $e(x_0, y)$, x_0 fixed, is just a line. The best possible counterstragey for player B is the value of y that minimizes e.

So there are only two possible (best) counter-strategies for B: either

$$C = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{or} \quad C = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

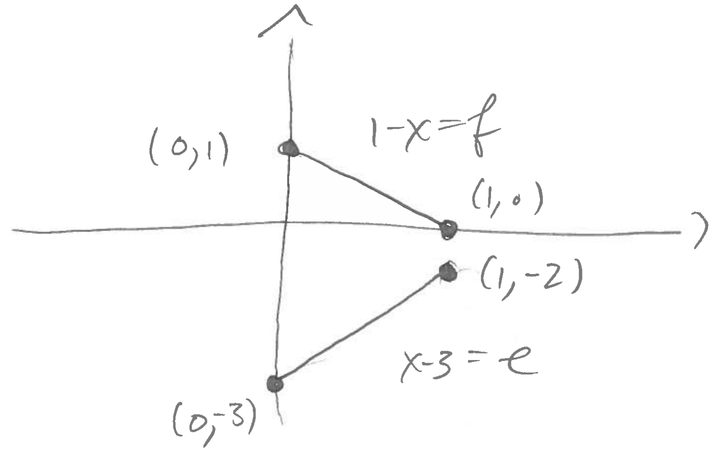
So, we need only compute the following scenarions:

$$e = \begin{bmatrix} x & 1-x \end{bmatrix} \begin{bmatrix} -2 & 0 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} x & 1-x \end{bmatrix} \begin{bmatrix} -2 \\ -3 \end{bmatrix} = -2x - 3(1-x) = -2x - 3 + 3x = \underline{1x - 3}$$

and

$$f = \begin{bmatrix} x & 1-x \end{bmatrix} \begin{bmatrix} -2 & 0 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} x & 1-x \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 1 - x.$$

These are both lines, on the same coordinate plane, they look like:



Take the strategy (x)=1, since this mitigates as much damage as possible.

To determine the optimal strategy for player B

compute

$$e = \underset{1\times 2}{\begin{bmatrix} 1 & 0 \end{bmatrix}} \underset{2\times 2}{\begin{bmatrix} -2 & 0 \\ -3 & 1 \end{bmatrix}} \begin{bmatrix} y \\ 1-y \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 0 \end{bmatrix} \begin{bmatrix} y \\ 1-y \end{bmatrix}$$

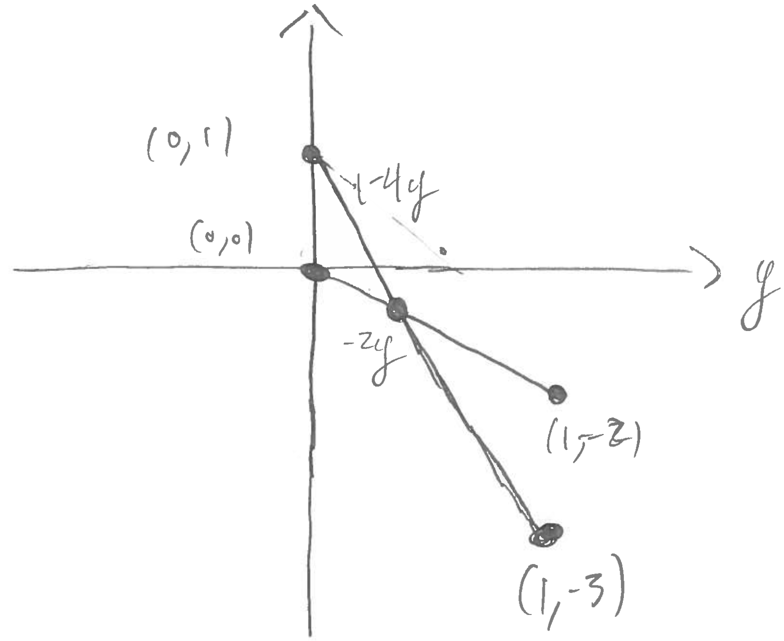
$$= \begin{bmatrix} -2 & 0 \end{bmatrix} \begin{bmatrix} y \\ 1-y \end{bmatrix} = -2y + 0(1-y) = -2y.$$

$$f = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} -2 & 0 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} y \\ 1-y \end{bmatrix}$$

$$= \begin{bmatrix} -3 & 1 \end{bmatrix} \begin{bmatrix} y \\ 1-y \end{bmatrix} = -3y + (1-y)$$

$$= -3y - y + 1$$

$$= -4y + 1.$$



To minimize player A's expected payout, we want to take $y=1$, which maximizes player B's expected payout.

Think about a function of one variable on a closed interval $[a,b]$. Say $f(x)=x^3-x-1$, on $[-1,1]$.

Solve:

$$f'(x) = 3x^2 - 1 = 0$$
$$3x^2 = 1$$
$$\Rightarrow x^2 = 1/3$$
$$\Rightarrow x = \pm\sqrt{1/3} = \pm 1/\sqrt{3}.$$

Plug $\pm 1, \pm 1/\sqrt{3}$ into $f(x)$, see which is biggest. This gives the maximum on $[-1,1]$.

$$P = \begin{bmatrix} -2 & 0 \\ 3 & -1 \end{bmatrix}$$

The maximization analogue for functions of 2 variables is

$R = [x \quad 1-x] \qquad C = [y \quad 1-y]$

$e = RPC = 4y - 6xy + x - 1$

$e_x = -6y + 1$

$e_{xx} = 0$

$e_y = 4 - 6x$

$e_{yy} = 0$

$e_{xy} = -6$

$e_{yx} = -6$

$$H = \begin{bmatrix} e_{xx} & e_{xy} \\ e_{yx} & e_{yy} \end{bmatrix} = \begin{bmatrix} 0 & -6 \\ -6 & 0 \end{bmatrix}$$

$D = \det(H) = 0 - (-6)^2 = -36.$