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If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $\det(A) = ad - bc \neq 0$

then

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Pf: Since $ad - bc \neq 0$, we may assume that $a \neq 0$. (Why? If $a = 0 = c$, then

$$ad - bc = 0 \cdot d - b \cdot 0 = 0$$

which we assumed not to be the case, so at least one of a, c is non-zero. If $a \neq 0$, done. If $c \neq 0$, interchange rows e.g.

$$\begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix})$$

Perform Gauss-Jordan Elimination on the matrix (augmented)

$$\left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right]$$

$$\xrightarrow{aR_2 - cR_1} \left[\begin{array}{cc|cc} a & b & 1 & 0 \\ a \cdot c - c \cdot a & ad - bc & -c & a \end{array}\right] = \left[\begin{array}{cc|cc} a & b & 1 & 0 \\ 0 & ad-bc & -c & a \end{array}\right]$$

$$(ad-bc)R_1 - bR_2 \quad \longrightarrow \left[\begin{array}{cc|cc} a(ad-bc) & (ad-bc)b - (ad-bc)b & (ad-bc) - b(-c) & -ba \\ 0 & ad-bc & -c & a \end{array}\right]$$

$$= \left[\begin{array}{cc|cc} a(ad-bc) & 0 & ad & -ba \\ 0 & ad-bc & -c & a \end{array}\right]$$

$$\xrightarrow[\frac{1}{ad-bc}R_2]{\frac{1}{a(ad-bc)}R_1} \left[\begin{array}{cc|cc} 1 & 0 & \frac{\cancel{a}d}{\cancel{a}(ad-bc)} & \frac{-b\cancel{a}}{\cancel{a}(ad-bc)} \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right]$$

$$= \left[\begin{array}{cc|cc} 1 & 0 & \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right]$$

$$\underbrace{\begin{matrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{matrix}}_{\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = A^{-1}.}$$

$\blacksquare$

4.4 Game Theory (p. 267)

Def^n: A two-person zero sum game is a game in which one players loss is the other's gain.

E.g.: Rock, paper, scissors.

Say the loser pays the winner \$1 after each round.

~~Def^n~~: We assume that each player can choose moves from a fixed, finite set.

Def^n: If player A has m moves and player B has n moves, we can represent the game using an m×n matrix called the payoff matrix showing the result of each possible pair of choices of moves.

E.g.:

$$\begin{array}{cc} & \quad B \\ & \begin{array}{ccc} r & p & s \end{array} \\ A \begin{array}{c} r \\ p \\ s \end{array} & \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix} \end{array}$$

This is the payoff matrix for the rock, paper, scissors game above.

Def^n: The way a player chooses to move is called a <u>strategy</u>.

- If a player always chooses the same move, this is a <u>pure</u> <u>strategy</u>.
- otherwise, a <u>mixed</u> <u>strategy</u>.

E.g. Player A mixed strategy: 50% rock
25% paper
25% scissors

Player B pure strategy: 0% rock
100% paper
0% scissors.

Expected Payoff

E.g.: Consider the game

$$
\begin{array}{ccc}
 & & B \\
 & a & b \\
A \quad p & \left[\begin{array}{c} 3 \\ -2 \end{array}\right. & \left.\begin{array}{c} -1 \\ 3 \end{array}\right] \\
\quad\ q & &
\end{array}
$$

Player A strategy: p $3/4$ of time (75%)
q $1/4$ of time (25%)

Player B strategy: a $1/5$ of time (20%)
b $4/5$ of time (80%)

Assume they play 100 games. On average, how much does A expect to win/lose?

Each game has 4 possible results:

$$\{p,q\} \times \{a,b\} = \{(p,\overset{a}{\cancel{b}}), (p,b), (q,a), (q,b)\}.$$

Case 1: (p,a) Player A should have played p $3/4$ of the time, or 75 times. Of the 75 times, player B played a $1/5$ of the time, or $75/5 = \frac{15 \cdot 5}{5} = 15$ times.

Easy Computation: $\left(\frac{3}{4}\right)\left(\frac{1}{5}\right)\cdot 100 = 15$

<u>Case 2:</u> (p,b) $100\left(\frac{3}{4}\right)\left(\frac{4}{5}\right) = \frac{3}{5}\cdot 100 = 60$

<u>Case 3:</u> (q,a) $\left(\frac{1}{4}\right)\left(\frac{1}{5}\right)\cdot 100 = \frac{100}{20} = 5$

<u>Case 4:</u> (q,b) $\left(\frac{1}{4}\right)\left(\frac{4}{5}\right)\cdot 100 = \frac{1}{5}\, 100 = 20.$