Matrix Inversion

Given an $n \times n$ (square) matrix

$$M = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

Form the augmented matrix

$$\left(\underbrace{\begin{matrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{matrix}}_{M} \;\middle|\; \underbrace{\overbrace{\begin{matrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{matrix}}^{n}}_{I_{n \times n}}\right) \left.\vphantom{\begin{matrix} 1 \\ 0 \\ \vdots \\ 0 \end{matrix}}\right\} n$$

If we can use the elementary row operations to obtain the augmented matrix

$$\left(\underbrace{\begin{matrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{matrix}}_{I_{n \times n}} \;\middle|\; \underbrace{\begin{matrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{matrix}}_{\text{This is } M^{-1}}\right)$$

Eg.:
$$-x+2y-z=0$$
$$-x-y+2z=0$$
$$2x+0y-z=4$$

Solution $(4,4,4)$

$$A=\begin{bmatrix}-1 & 2 & -1\\ -1 & -1 & 2\\ 2 & 0 & -1\end{bmatrix}$$

Want to find A^{-1}.

$$\left(\begin{array}{ccc|ccc}-1 & 2 & -1 & 1 & 0 & 0\\ -1 & -1 & 2 & 0 & 1 & 0\\ 2 & 0 & -1 & 0 & 0 & 1\end{array}\right)\xrightarrow[R_3+2R_1]{R_2-R_1}$$

$$\left(\begin{array}{ccc|ccc}-1 & 2 & -1 & 1 & 0 & 0\\ 0 & -3 & 3 & -1 & 1 & 0\\ 0 & 4 & -3 & 2 & 0 & 1\end{array}\right)\xrightarrow[3R_3+4R_2]{3R_1+2R_2}$$

$$\left(\begin{array}{ccc|ccc}-3 & 0 & 3 & 1 & 2 & 0\\ 0 & -3 & 3 & -1 & 1 & 0\\ 0 & 0 & 3 & 2 & 4 & 3\end{array}\right)\xrightarrow[R_2-R_3]{R_1-R_3}$$

$$\left(\begin{array}{ccc|ccc}-3 & 0 & 0 & -1 & -2 & -3\\ 0 & -3 & 0 & -3 & -3 & -3\\ 0 & 0 & 3 & 2 & 4 & 3\end{array}\right)$$

$$\xrightarrow[\substack{-\frac{1}{3}R_2\\ \frac{1}{3}R_3}]{-\frac{1}{3}R_1}\left(\begin{array}{ccc|ccc}1 & 0 & 0 & \frac{1}{3} & \frac{2}{3} & 1\\ 0 & 1 & 0 & 1 & 1 & 1\\ 0 & 0 & 1 & 2/3 & 4/3 & 1\end{array}\right)$$

$$A^{-1} = \begin{bmatrix}1/3 & 2/3 & 1\\ 1 & 1 & 1\\ 2/3 & 4/3 & 1\end{bmatrix}$$

$$-x + 2y - z = 1$$
$$-x - y + 2z = 2$$
$$2x + 0y - z = 4$$

$$A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$$

$$A^{-1}A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1}\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1/3 & 2/3 & 1 \\ 1 & 1 & 1 \\ 2/3 & 4/3 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 1/3 + 4/3 + 12/3 \\ 7 \\ 2/3 + 8/3 + 12/3 \end{bmatrix}$$

$$= \begin{bmatrix} 17/3 \\ 7 \\ 22/3 \end{bmatrix}.$$

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§3.3 Applications of Systems of Linear Equations

E.g.: (1) Pine Orange : 2q pineapple, 2q orange

Pine Kiwi : 3q Pineapple, 1q Kiwi

Orange Kiwi : 3q orange, 1q kiwi

Each day have 800 q pineapple, 650 q orange, 350 q Kiwi. How many gallons of each blend to use all materials?

Let x = # gallons Pine Orange

y = # gallons Pine Kiwi

z = # gallons Orange Kiwi.

$$2x + 3y + 0z = 800$$ (Quarts Pineapple)

$$2x + 0y + 3z = 650$$ (Quarts Orange)

$$0 \cdot x + 1 \cdot y + 1 \cdot z = 350$$ (Quarts Kiwi)

$$\left[\begin{array}{ccc|c} 2 & 3 & 0 & 800 \\ 2 & 0 & 3 & 650 \\ 0 & 1 & 1 & 350 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 100 \\ 0 & 1 & 0 & 200 \\ 0 & 0 & 1 & 150 \end{array}\right]$$

100 gallons Pine Orange

200 gallons Pine Kiwi

150 gallons Orange Kiwi.