Def^{n}: If M is a matrix and a is any number, the product $a \cdot M$ is called scalar multiplication / multiplication by the scalar a and is defined to be the matrix obtained by multiplying the entries of M by a.

E.g.: $M = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $\quad a = 3$

$$a \cdot M = 3 \cdot M = \begin{bmatrix} 3\cdot 1 & 3\cdot 2 \\ 3\cdot 3 & 3\cdot 4 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 9 & 12 \end{bmatrix}$$

In fact, if we're interested in $n \times n$ matrices, we can think about numbers as $n \times n$ matrices. If a is any real number,

$$a \cdot I_{n\times n} = a \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} = \begin{bmatrix} a & 0 & 0 & \cdots & 0 \\ 0 & a & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a \end{bmatrix}$$

E.g: $n=2$, $a=3$, M as above

$$3 \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 3\cdot 1 + 0\cdot 3 & 3\cdot 2 + 0\cdot 4 \\ 0\cdot 1 + 3\cdot 3 & 0\cdot 2 + 3\cdot 4 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 6 \\ 9 & 12 \end{bmatrix} = 3 \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \end{bmatrix} \begin{bmatrix} b_{11} \\ b_{21} \\ \vdots \\ b_{n1} \end{bmatrix} = a_{11} b_{11} + a_{12} b_{21} + \dots a_{1n} b_{n1}.$$

Recall: $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ the determinant of this matrix is $a_{11} \cdot a_{22} - a_{12} \cdot a_{21}$

and has an inverse

$$\frac{1}{a_{11} a_{22} - a_{12} \cdot a_{21}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} = A^{-1}$$

if and only if $a_{11} a_{22} - a_{12} \cdot a_{21} \neq 0$.

$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $\quad 4\cdot 1 - 2\cdot 3 = 4-6 = -2 \neq 0$

$$A^{-1} = \frac{1}{-2}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 3/2 & \frac{-1}{2} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} -2+3 & 1+-1 \\ -6+6 & 3-2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_{2\times 2}.$$

Gauss - Jorden Reduction

Elementary Row Operations

Type 1: Replace a row by a non-zero multiple of that row.

Type 2: Replace a row, R_i, by $aR_i \pm bR_j$, $a, b \neq 0$.

Type 3: Switch the order of the rows.

Perform these operations on an "augmented matrix"

If we have a system of n equations in unknowns/variables $x_1, x_2, \dots, x_n$

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_{11} \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_{21} \\
\vdots \qquad\quad \vdots \qquad\qquad\quad &\vdots \\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_{n1}
\end{aligned}
$$

the Augmented Matrix

$$
\left[\begin{array}{cccc:c}
a_{11} & a_{12} & \cdots & a_{1n} & b_{11} \\
a_{21} & a_{22} & \cdots & a_{2n} & b_{21} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{nn} & b_{n1}
\end{array}\right]
$$

This is short-hand for our usual matrix equation.

E.g.:
$$
\begin{aligned}
x + 2y &= 5 \\
3x + 4y &= 6
\end{aligned}
\qquad
\underbrace{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}}_{\text{usual matrix equation.}}
$$

Augmented Matrix

$$
\left[\begin{array}{cc:c}
1 & 2 & 5 \\
3 & 4 & 6
\end{array}\right]
$$

The Augmented Matrix is used to find a solution to a system in the following way:

Apply the elementary row operations until the augmented matrix has the $n \times n$ identity on the left hand side

$$\left[\begin{array}{c|c} \underbrace{\begin{matrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{matrix}}_{I_{n \times n}} & \begin{matrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{matrix} \end{array}\right]$$

then the solution to the system is

$$x_1 = s_1$$
$$x_2 = s_2$$
$$\vdots$$
$$x_n = s_n .$$

<u>E.g.</u>: $-\frac{2}{3}x + \frac{1}{2}y = -3$

$\frac{1}{4}x - y = \frac{11}{4}$

(p. 191/192)

$$\left[\begin{array}{cc|c} -2/3 & \frac{1}{2} & -3 \\ \frac{1}{4} & -1 & \frac{11}{4} \end{array}\right]$$

<u>Step 0</u>: Check the determinant.

$-2/3(-1) - \frac{1}{2}\frac{1}{4} = \frac{2}{3} - \frac{1}{8} \neq 0.$

Step 1: Clear fractions

$$\left[\begin{array}{cc|c} -2/3 & \frac{1}{2} & -3 \\ \frac{1}{4} & -1 & \frac{11}{4} \end{array}\right] \xrightarrow[4 \cdot R_2]{6 \cdot R_1} \left[\begin{array}{cc|c} -4 & 3 & -18 \\ 1 & -4 & 11 \end{array}\right]$$

Step 2: Designate the first non-zero entry in the first row as the "pivot" - use this to clear the column.

Replace R_2 by $4 \cdot R_2 + R_1$

$$\left[\begin{array}{cc|c} -4 & 3 & -18 \\ 1 & -4 & 11 \end{array}\right] \xrightarrow{4R_2 + R_1}$$

$4R_2 + R_1$

$$\left[\begin{array}{cc|c} -4 & 3 & -18 \\ 4 \cdot 1 + (-4) & 4(-4) + 3 & 4(11) + (-18) \end{array}\right]$$

$$\begin{array}{r} 44 \\ -18 \\ \hline 26 \end{array}$$

$$= \left[\begin{array}{cc|c} -4 & 3 & -18 \\ 0 & -13 & 26 \end{array}\right] \xrightarrow{\frac{1}{13} R_2} \left[\begin{array}{cc|c} -4 & 3 & -18 \\ 0 & -1 & 2 \end{array}\right]$$