$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}$$

$$(2\times 2)(2\times 2) = (2\times 2)$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{bmatrix}$$

$$(m\times n)(n\times k) = (m\times k)$$

$$(2\times 2)(2\times 3) = (2\times 3)$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} - \\ - \end{bmatrix}$$

$$(2\times 2)\ (2\times 1) = (2\times 1)$$

$$\begin{bmatrix} x \\ y \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$ ← Cannot multiply these together

$$(2\times 1) \cdot (2\times 2)$$

System of Equations $\leftrightarrow$ Matrix Equations

Eg: $a_{11}x + a_{12}y = b_{11}$

$a_{21}x + a_{22}y = b_{21}$

$$\longleftrightarrow \underbrace{\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\underline{X}} = \underbrace{\begin{bmatrix} b_{11} \\ b_{21} \end{bmatrix}}_{B}$$

Want to know how to solve

$$A\underline{X} = B$$

If we have the inverse of A, A^{-1}, then

$$A^{-1}A = I \quad \text{(by definition)}$$

$$A^{-1}(A\underline{X}) = A^{-1}B$$

$$\Rightarrow I\underline{X} = A^{-1}B$$

$$\Rightarrow \underline{X} = A^{-1}B.$$

How do we find A^{-1}?

Eg: $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ We want to find some matrix

$$A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Such that

$$A^{-1}A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_{2\times 2}$$

$$A^{-1}A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} a+b & a\cdot 0+b \\ c+d & c\cdot 0+d \end{bmatrix}$$

$$= \begin{bmatrix} a+b & b \\ c+d & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$a+b = 1 \qquad b = 0$

$c+d = 0 \qquad d = 1$

$$A^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Eg:

$$\left.\begin{aligned} 1\cdot x + 0\cdot y &= 5 \\ x &= 5 \\ x+y &= 7 \end{aligned}\right\} \text{system}$$

Matrix Equation:

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 7 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5+0\cdot 7 \\ -5+7 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

$(5,2)$ is the solution to the system.

Given a matrix

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

the inverse of M is

$$\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix}$$

provided that $ad-bc \neq 0$.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} ad-bc & -ab+ab \\ cd-cd & ad-cb \end{bmatrix}$$

$$= \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix}$$

Def^n: The determinant of the matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is $ad-bc$.

Thm: The matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ has an inverse if and only if $ad-bc \neq 0$. The inverse is

$$\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Ex.: $2x + 3y = 2$

$-x - \frac{3}{2}y = -\frac{1}{2}$.

$$\begin{bmatrix} 2 & 3 \\ -1 & -3/2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ -\frac{1}{2} \end{bmatrix}$$

$$2(-3/2) - (3)(-1) = -3 + 3 = 0$$

No Solutions!

$$2x + 3y = 2 \Rightarrow 3y = -2x + 2$$
$$\Rightarrow y = \frac{-2}{3}x + \frac{2}{3}$$

$$-x - \frac{3}{2}y = \frac{-1}{2} \Rightarrow \frac{3}{2}y = -x + \frac{1}{2}$$
$$\rightarrow y = \frac{-2}{3}x + \frac{2}{6}.$$

E.g.: $2x - 3y = 2$

$6x - 9y = 3$

$$\begin{bmatrix} 2 & -3 \\ 6 & -9 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$2(-9) - (-3)(6) = -18 + 18 = 0.$$

No solutions.

E.g.: $3x - 2y = 6$

$2x - 3y = -6$

$$A \to \begin{bmatrix} 3 & -2 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ -6 \end{bmatrix}$$

$$3(-3) - (-2)(2) = -9 + 4 = -5 \neq 0!$$

$$A^{-1} = \begin{bmatrix} \frac{-3}{-5} & \frac{2}{-5} \\ -\cancel{2/5} & \cancel{3/5} \end{bmatrix} = \begin{bmatrix} 3/5 & -2/5 \\ 2/5 & -3/5 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 3/5 & -2/5 \\ 2/5 & -3/5 \end{bmatrix} \begin{bmatrix} 6 \\ -6 \end{bmatrix} = \begin{bmatrix} \frac{18}{5} + \frac{12}{5} \\ 12/5 + \frac{18}{5} \end{bmatrix}$$

$$= \begin{bmatrix} 6 \\ 6 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

Solution: $(6, 6)$