Def^n: An $m \times n$-matrix is a rectangular array of numbers, called entries, with m rows and n columns

$$
\begin{array}{cc}
 & \overbrace{\hphantom{a_{11} \quad a_{12} \quad a_{13} \quad \cdots \quad a_{1n}}}^{n \text{ columns}} \\
m \text{ rows} \left\{ \vphantom{\begin{matrix} a \\ a \\ a \\ \vdots \\ a \end{matrix}} \right. & \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{pmatrix}
\end{array}
$$

Eg.: 2×2 square $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ | (1×3) $\begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$ row matrix | (2×1) $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ column matrix

2×3 $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$

3×2 $\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}$

If $m=n$, say the matrix square

If $m=1$, the matrix is called a row matrix/vector

If $n=1$, the matrix is called a column matrix/vect

Def^n: Say two matrices M and N are equal if they have the same entries in the same order, and the same dimensions.

Eg: $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \neq \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$

$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \iff x = 1$ and $y = 2$.

$\begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \iff x = 1,\ y = 0,\ z = 1,\ w = 1.$

Addition & Subtraction

Given two $m \times n$ matrices, M and N, we can add and subtract by adding the entries. The result is also an $m \times n$ matrix

$$M = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}, \quad N = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{pmatrix}$$

$$M \pm N = \begin{pmatrix} a_{11} \pm b_{11} & a_{12} \pm b_{12} & \cdots & a_{1n} \pm b_{1n} \\ a_{21} \pm b_{21} & a_{22} \pm b_{22} & \cdots & a_{2n} \pm b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} \pm b_{m1} & a_{m2} \pm b_{m2} & \cdots & a_{mn} \pm b_{mn} \end{pmatrix}$$

Eg:

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} 4 & 5 \\ 6 & 7 \end{pmatrix} = \begin{pmatrix} 1+4 & 2+5 \\ 3+6 & 4+7 \end{pmatrix} = \begin{pmatrix} 5 & 7 \\ 9 & 11 \end{pmatrix}$$

Eg:

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} - \begin{pmatrix} 10 & 11 & 12 \\ 13 & 14 & 15 \\ 16 & 17 & 18 \end{pmatrix} = \begin{pmatrix} -9 & -9 & -9 \\ -9 & -9 & -9 \\ -9 & -9 & -9 \end{pmatrix}$$

Eg: Auto parts store, one in Vancouver, one in Quebec,

Jan

	Van	Quebec
wiper blades	20	15
cleaning fluid	10	12
floor mats	8	4

Feb

Van	Quebec
23	12
8	12
4	5

Calculate the change in sales for both locations.

$$\begin{pmatrix} 23 & 12 \\ 8 & 12 \\ 4 & 5 \end{pmatrix} - \begin{pmatrix} 20 & 15 \\ 10 & 12 \\ 8 & 4 \end{pmatrix} = \begin{pmatrix} 3 & -3 \\ -2 & 0 \\ -4 & 1 \end{pmatrix}$$

Matrix Multiplication

Def$^{\underline{n}}$: Given a row matrix of dimension $1 \times n$

$$M = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \end{pmatrix}$$

and a column matrix of dimension $n \times 1$

$$N = \begin{pmatrix} b_{11} \\ b_{21} \\ \vdots \\ b_{n1} \end{pmatrix}$$

then we can define their product to be

$$M \cdot N = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \end{pmatrix} \begin{pmatrix} b_{11} \\ b_{21} \\ \vdots \\ b_{n1} \end{pmatrix}$$

$$= a_{11} b_{11} + a_{12} b_{21} + \cdots + a_{1n} b_{n1}.$$

Eg: $\begin{pmatrix} 2 & 4 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 10 \\ -1 \end{pmatrix} = 2 \cdot 2 + 4 \cdot 10 + 1(-1)$

$= 4 + 40 - 1 = 43.$

Eg: Auto parts store, in January sold the following items

	V	Q
W.B.	20	15
C.F.	10	12
F.M.	8	4

W.B cost \$7 ea., bottles of C.F. cost \$3 ea., F.M. cost \$12 ea.

How much money did each location make?

Vancouver:

$$\begin{pmatrix} 7 & 3 & 12 \end{pmatrix} \begin{pmatrix} 20 \\ 10 \\ 8 \end{pmatrix} = 7\cdot 20 + 3\cdot 10 + 12\cdot 8$$
$$= 140 + 30 + 96$$
$$= 266.$$

<u>Quebec</u>

$$\begin{pmatrix} 7 & 3 & 12 \end{pmatrix} \begin{pmatrix} 15 \\ 12 \\ 4 \end{pmatrix} = 7\cdot 15 + 3\cdot 12 + 12\cdot 4$$
$$= 105 + 36 + 48$$
$$= 141 + 48$$
$$= 189.$$

Eg: Linear equation in 4 variables:

$$3x + y - z + 2w = 8$$

We can represent this as a matrix equation:

$$\begin{pmatrix} 3 & 1 & -1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = 8.$$

coefficient matrix.

Def^{n}: Given two matrices

$$M - m \times n$$

$$N - n \times k$$

the product of M and N is an $m \times k$-matrix.

$$M = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

$$MN = \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1k} \\ c_{21} & c_{22} & \cdots & c_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mk} \end{pmatrix}$$

$$N = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1k} \\ b_{21} & b_{22} & \cdots & b_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & \cdots & \cdots & b_{nk} \end{pmatrix}$$

Where

$$c_{ij} = \begin{pmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \end{pmatrix} \begin{pmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{pmatrix}$$

Eg:

$$\begin{pmatrix} 2 & 1 \\ 5 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x + y \\ 5x + y \end{pmatrix}$$