$$(x+y)^2 = (x+y)(x+y) = xx + xy + yx + yy$$

$$= x^2 + xy + xy + y^2$$

$$= x^2 + 2xy + y^2$$

Rmk: $xy + xy = xy(1+1) = \cancel{xy} \; xy\,2 = 2xy.$

What about

$(x+y)^3$?

$$(x+y)^3 = (x+y)(x+y)(x+y)$$

$$= (x+y)^2(x+y)$$

$$= (x^2 + 2xy + y^2)(x+y)$$

$$= xx^2 + x2xy + xy^2 + yx^2 + y2xy + yy^{\cancel{3}2}$$

$$= x^3 + \boxed{2x^2y} + xy^2 + \boxed{x^2y} + 2xy^2 + y^3$$

$$= x^3 + 3x^2y + 3xy^2 + y^3.$$

This is awful.

Another way to expand this

$$\cancel{(x+y)(x+y)(x+y)} =$$

$$(x+y)(x+y)(x+y) = xxx + xxy + xxy$$
$$+ xyy + yxx + yxy$$
$$+ yyx + yyy$$
$$= x^3 + \boxed{x^2y} + \boxed{x^2y} + xy^2$$
$$+ \boxed{x^2y} + xy^2 + xy^2 + y^3$$
$$= x^3 + 3x^2y + 3xy^2 + y^3$$
$$= x^3y^0 + 3x^2y^1 + 3xy^2 + x^0y^3$$

This is also awful; possibly worse.

In each monomial, x^iy^j ~~wt~~ we have $i+j=3$.

For general n, the monomials in the expansion of $(x+y)^n$ all have the form

$$x^{n-i}y^i, \text{ for } i \in \{0,1,2,3,\dots,n\}.$$

When expanding

$$(x+y)^n = \underbrace{(x+y)(x+y)\cdots(x+y)}$$

how many ways are there to form a monomial

$$x^{n-i}y^i$$

for each i? This is equivalent to counting how many ways to choose i y's we can choose from a set of n y's. This is

$$C(n,i) = \binom{n}{i} \quad \text{"n choose i."}$$

Thm (Binomial Theorem):

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \dots + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n$$

E.g.: $(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$

Thm (Pascal's Identity): If n, k are positive integers, ~~then~~ and $k \le n$, then

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}.$$

Pf: $$\frac{n!}{(n-(k-1))!\,(k-1)!} + \frac{n!}{(n-k)!\,k!} = n!\left(\frac{1}{(n-k+1)!\,(k-1)!} + \frac{1}{(n-k)!\,k!}\right)$$

$$= n!\left(\frac{k}{(n-k+1)!\,(k-1)!\,k} + \frac{n-k+1}{(n-k+1)(n-k)!\,k!}\right)$$

$$= n!\left(\frac{k}{(n-k+1)!\,k!} + \frac{n-k+1}{(n-k+1)!\,k!}\right)$$

$$= n!\left(\frac{\cancel{k}+n-\cancel{k}+1}{(n-k+1)!\,k!}\right)$$

$$= n!\left(\frac{n+1}{(n-k+1)!\,k!}\right)$$

$$= \frac{(n+1)!}{((n+1)-k)!\,k!}$$

$$= \binom{n+1}{k}. \blacksquare$$

$$(n-k)! = (n-k)(n-(k-1))(n-(k-2))\cdots 2\cdot 1$$

$$(n-k+1)! = (n-k+1)((n-k+1)-1)((n-k+1)-2)\cdots 2\cdot 1$$

$$= (n-k+1)\underbrace{(n-k)(n-k-1)\cdots 2\cdot 1}_{(n-k)!}$$

Eg: $n = 5$ $\quad (n-k)! = (5-2)! = 3!$

$k = 2$ $\quad (n-k+1)! = (5-2+1)! = 4!$

Pascal's Triangle

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

$$(x+y)^6 = x^6 + 6x^5y + \overset{15}{\cancel{20}}x^4y^2 + \overset{20}{\cancel{15}}x^3y^3 + \overset{15}{\cancel{6}}x^2y^4 + 6xy^5 + y^6.$$