Def^n 2/1/16 ①

A permutation of n elements taken r at a time (r-permutation) is an ordered list of r elements taken from a set of n elements.

E.g: $\{a, b, c, d\}$ a 3-permutation is an ordered list of 3 of these letters. Some 3-permutations are

bad cad
dab bcd etc
cab dcb

How many 3-letter words can we make with these four letters? There are

- 4 ways to choose the first letter
- 3 ways to choose the second letter
- 2 ways to choose the third letter

$4 \cdot 3 \cdot 2 = 24$ possible words (3 letters long)

How many two letter words can we make with these four letters? There

- 4 choices for the first letter
- 3 choices for the second letter

There are $4 \cdot 3 = 12$ possible words (2 letters long).

For an r-permutation of n elements there are

1 • n choices for the first object,

2 • $n-1$ choices for the second,
$n-(2-1)$

3 • $n-2$ choices for the third
$n-(3-1)$

⋮

$r-1$ • $n-(r-1-1) = n-(r-2) = n-r+2$ ~~choices~~ choices for the $r-1^{st}$ object

r • $n-(r-1) = n-r+1$ choices for the r^{th} object

So there are

$$n \cdot (n-1)(n-2) \cdots (n-r+2)(n-r+1)$$

r-permutations.

The reason why there are

$$n-r+1 = n-(r-1)$$

objects to choose from at the r^{th} step is because there are $r-1$ steps that have occurred previously and on each of those $r-1$ steps we have removed one object.

The number of r-permutations of a set with n elements is

$$P(n,r) = n(n-1)(n-2)\cdots(n-r+1)$$

$$= n(n-1)(n-2)\cdots(n-r+1)\left(\frac{(n-r)(n-r-1)\cdots(2)(1)}{(n-r)(n-r-1)\cdots(2)(1)}\right)$$

$$= \frac{n!}{(n-r)!}$$

E.g.: $\{a, b, c, d\}$ $\quad P(4,3) = \frac{4!}{(4-3)!} = \frac{4!}{1!} = 4\cdot 3\cdot 2\cdot 1 = 24$

$$P(4,2) = \frac{4!}{(4-2)!} = \frac{4!}{2!} = \frac{24}{2} = 12.$$

Given 6 distinct letters, how many 3-letter (4) words are possible?

This is the number of 3-permutations of a set of 6 elements:

$$P(6,3) = \frac{6!}{(6-3)!} = \frac{6!}{3!} = \frac{6\cdot5\cdot4\cdot\cancel{3}\cdot\cancel{2}\cdot\cancel{1}}{\cancel{3}\cdot\cancel{2}\cdot\cancel{1}} = 120$$

E.g.: Say there five contestants on a game show which awards three prizes. How many ways the prizes be distributed?

$$P(5,3) = \frac{5!}{(5-3)!} = \frac{5!}{2!} = 5\cdot4\cdot3 = 60.$$

~~Defn~~ Rmk: A permutation of a set of n elements is an n-permutation.

$$P(n,n) = \frac{n!}{(n-n)!} = \frac{n!}{0!} = \frac{n!}{1} = n!.$$

Def^n: A combination of n elements taken r at a time (r-combination) is an unordered set of r elements taken from a set of n elements.

E.g.: $\{a, b, c, d, e\}$

A ~~two~~ 2-combination could be

$\{a,b\}$ or $\{a,c\}$ or $\{a,d\}$ or $\{a,e\}$

$\{b,c\}$ or $\{b,d\}$ or $\{b,e\}$

etc.

How many r-combinations of a set with n elements are there?

Let $C(n,r)$ be the number of r-combinations.

For each set of r elements there are $P(r,r)$ ~~combinations~~ permutations. The number of r-permutations of the original set is

$$P(n,r) = C(n,r) \cdot P(r,r)$$

$$\Rightarrow C(n,r) = \frac{P(n,r)}{P(r,r)} = \frac{n!}{(n-r)!\, r!}$$