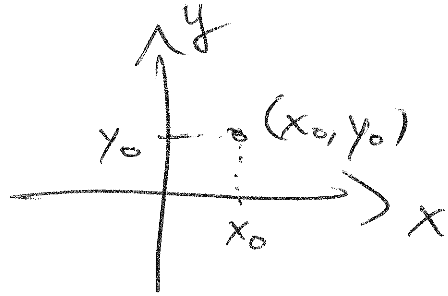
Def^{n}: The <u>Cartesian product</u> of two sets A and B is the set of all pairs of elements from A and B

$$A \times B = \{(a,b) \mid a \in A, b \in B\}.$$

E.g.: $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{(x,y) \mid x \in \mathbb{R}, y \in \mathbb{R}\}$

$\boxed{2 \times 5 = 10}$

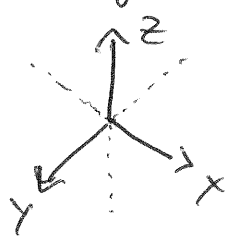
Cartesian plane



In general, given any n sets $S_1, S_2, S_3, \dots, S_n$ the Cartesian ~~of~~ product of these sets is the set of all n-tuples

$$S_1 \times S_2 \times S_3 \times \cdots \times S_n = \{(s_1, s_2, \dots, s_n) \mid s_1 \in S_1, s_2 \in S_2, \dots, s_n \in S_n\}$$

E.g.: $\mathbb{R}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R} = \{(x,y,z) \mid x \in \mathbb{R}, y \in \mathbb{R}, z \in \mathbb{R}\}$



Eg: $A = \{1, 2, 3\}$, $B = \{4, 5\}$

$$A \times B = \left\{ \begin{matrix} (1,4), & (2,4), & (3,4) \\ (1,5), & (2,5), & (3,5) \end{matrix} \right\}$$

E.g.: $A = \{\text{cow}, \text{dog}\}$ $\quad B = \{\text{pen}, \text{house}\}$

$$\begin{aligned} A \times B &= \{(a,b) \mid a \in A,\ b \in B\} \\ &= \{(\text{cow}, \text{pen}), (\text{cow}, \text{house}), \\ &\quad (\text{dog}, \text{pen}), (\text{dog}, \text{house})\}. \end{aligned}$$

Cardinality

If A and B are finite sets, the cardinality of $A \times B$ is

$$|A \times B| = |A| \cdot |B|.$$

Pf: Say $m = |A|$, $n = |B|$. Let $a_1, a_2, \ldots, a_m$ be the elements of A and let $b_1, b_2, \ldots, b_n$ be the elements of B.

We can form the cartesian product by the following method:

	b_1	b_2	b_3	$\cdots$	b_n
a_1	(a_1,b_1)	(a_1,b_2)	(a_1,b_3)	$\cdots$	(a_1,b_n)
a_2	(a_2,b_1)	(a_2,b_2)	(a_2,b_3)	$\cdots$	(a_2,b_n)
a_3	(a_3,b_1)	(a_3,b_2)	(a_3,b_3)	$\cdots$	(a_3,b_n)
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
a_m	(a_m,b_1)	(a_m,b_2)	(a_m,b_3)	$\cdots$	(a_m,b_n)

So there are

$$m \cdot n = |A| \cdot |B|$$

elements in the Cartesian Product. Therefore

$$|A \times B| = |A| \cdot |B|. \quad \blacksquare$$

<u>E.g.:</u> $A = \{1, 2, 3\}$, $B = \{1, 2, 3\}$

	1	2	3
1	(1,1)	(1,2)	(1,3)
2	(2,1)	(2,2)	(2,3)
3	(3,1)	(3,2)	(3,3)

Cardinality of Union

Eg.: $A = \{1,2,3\}$, $B = \{3,4,5\}$.

$$A \cup B = \{1,2,3,4,5\}.$$

$|A \cup B| = 5$, $|A| = |B| = 3$;

$$|A| + |B| = 6 \neq 5 = |A \cup B|.$$

Eg.: $A = \{1,2,3\}$, $B = \{4,5,6\}$

$$A \cup B = \{1,2,3,4,5,6\}$$

$$6 = |A \cup B| = |A| + |B| = 3+3.$$

The cardinality of the union is

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

When A and B are disjoint (i.e. $A \cap B = \emptyset$), then

$$\begin{aligned} |A \cup B| &= |A| + |B| - |A \cap B| \\ &= |A| + |B| - |\emptyset| \\ &= |A| + |B| - 0 = |A| + |B|. \end{aligned}$$

Eg: $A=\{1,2,3\}$, $B=\{2,3,4\}$

$A \cap B = \{2,3\}$

$|A \cap B| = 2$

$$|A \cup B| = |A| + |B| - |A \cap B|$$
$$= 3 + 3 - 2$$
$$= 4.$$

$A \cup B = \{1,2,3,4\}$.

Eg: (Disjoint Sets)

$A=\{1,2,3\}$, $B=\{4,5,6,7,8\}$.

$A \cap B = \emptyset$.

<u>Cardinality of the Complement</u>

Given $S \subseteq T$. The cardinality of $T \setminus S$ is

$$|T \setminus S| = |T| - |S|$$

Provided S and T are both <u>finite</u>.