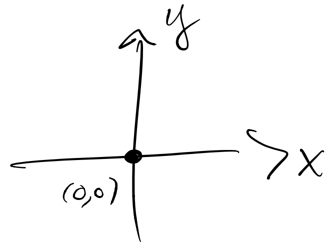
4/24/18

Polar Coordinates

Usual Cartesian plane with Cartesian coordinates

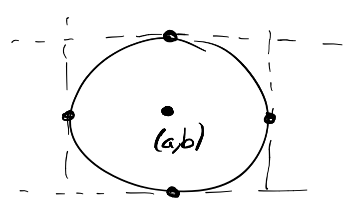


Some potential "not nice" things about this coordinate system:

Circles don't define functions in either variable

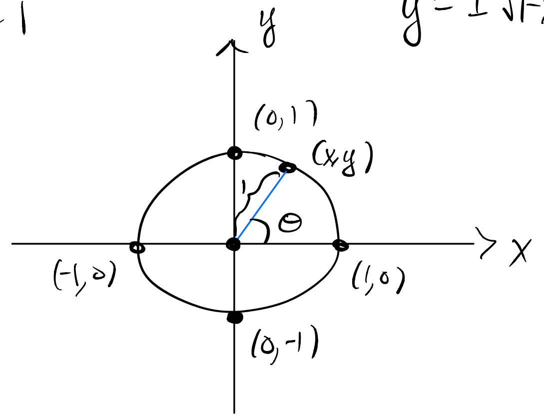
$$(x-a)^2 + (y-b)^2 = r^2$$

Any horiz/vertical line that intersects this graph:



intersects in 2 points, except at the tangent lines above.

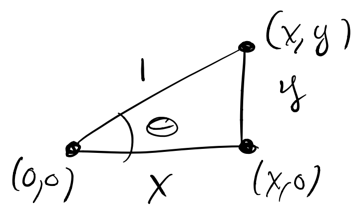
E.g.: $x^2+y^2=1$ $\qquad$ $y=\pm\sqrt{1-x^2}$ or $x=\pm\sqrt{1-y^2}$



Recall that any point on the circle has coordinates

$$x=\cos(\theta), \quad y=\sin(\theta)$$

for θ as below



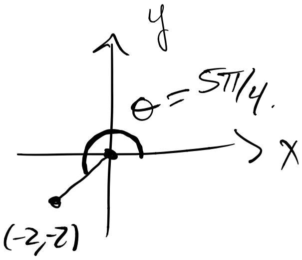
Any point on $x^2+y^2=1$ is determined by the equation

$$\tan(\theta) = \frac{y}{x}$$

Given θ, $x=\cos(\theta)$, $y=\sin(\theta)$.

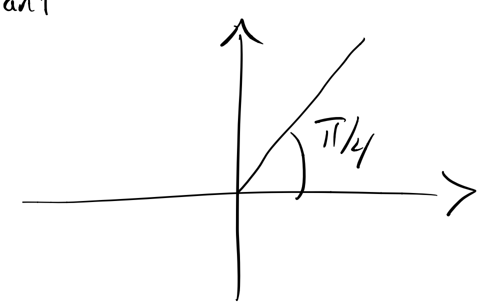
Given (x,y), $\theta = \arctan\left(\frac{y}{x}\right)$. Since arctan has range $(-\pi/2, \pi/2)$, sometimes you get a negative angle, but you can make this positive by adding 2π.

E.g.: $(-2,-2)$



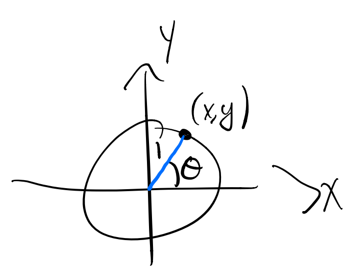
$\arctan\left(\frac{-2}{-2}\right) = \arctan(1) = $ either $\pi/4$ or $\frac{5\pi}{4}$

since $\pi/4$ corresponds to the point in the 1st quadrant



$\tan\left(\frac{-2}{-2}\right)$ must be $5\pi/4$.

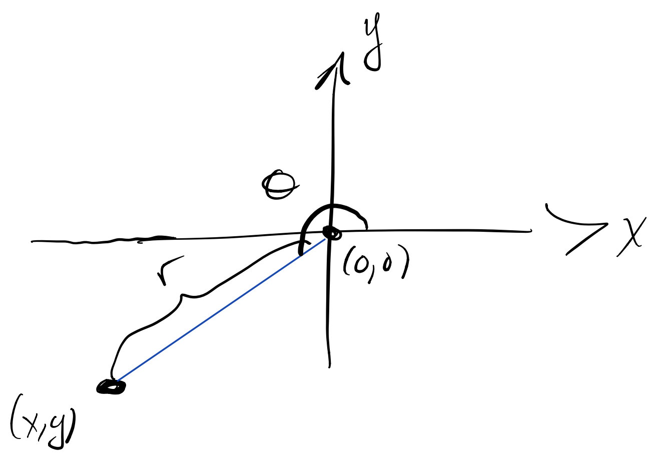
For points on the unit circle, we can represent (x,y) as $P(1,\theta)$, where in the picture



θ is the angle between x-axis & the blue line
1 is the length of

the blue line.

In general, for an arbitrary point in the plane (x, y), we want to mimick this construction.



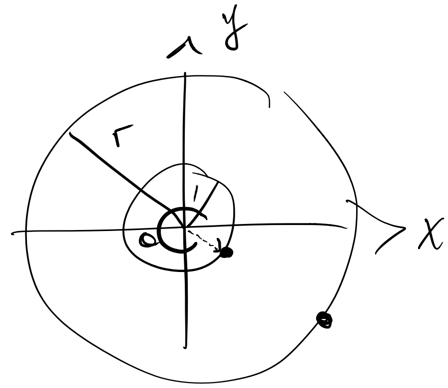
Assume that this point lies on a circle of radius r centered about the origin, so

$$x^2 + y^2 = r^2$$

$$\Rightarrow r = \sqrt{x^2 + y^2}$$

Also have an angle analogous to the unit circle case

$$\theta = \begin{cases} \tan\left(\frac{y}{x}\right), & x \neq 0 \\ \pi/2 & x = 0,\ y \geq 0 \\ -\pi/2 = 3\pi/2 & x = 0,\ y < 0. \end{cases}$$



To locate a point in the plane, we just need a radial distance, r, and an angle θ. We call this coordinate system "Polar Coordinates," the book decorates with a P, $P(r, \theta)$.

Effectively, these are parametric equations

$$x = r\cos(\theta)$$
$$y = r\sin(\theta).$$

$$\begin{aligned} x^2 + y^2 &= r^2\cos^2(\theta) + r^2\sin^2(\theta) \\ &= r^2(\cos^2(\theta) + \sin^2(\theta)) \\ &= r^2. \end{aligned}$$

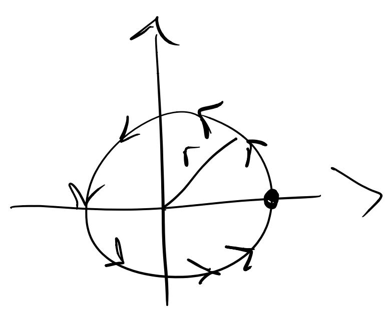
In polar coordinates the equation of a circle of radius r_0 is

$$r = r_0.$$

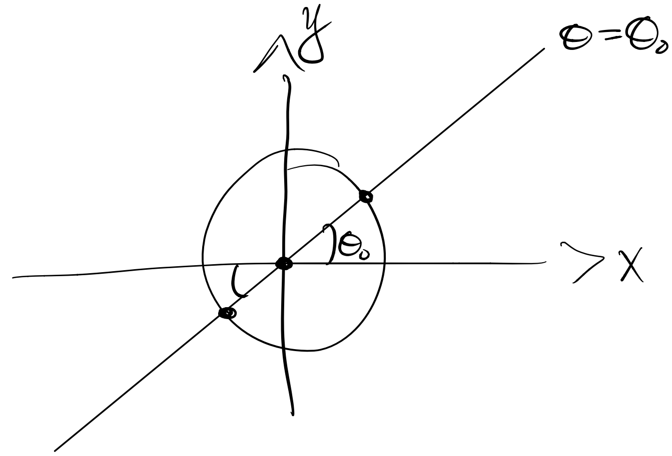
The set of all points (r,θ) satisfying this equation (i.e. the graph) is

$$\{(r,\theta) \mid r = r_0\} = \{(r_0,\theta) \mid \text{for all } \theta\}$$
$$= \{(r_0\cos(\theta), r_0\sin(\theta))\}$$
$$= \{(x,y) \mid x^2+y^2 = r_0^2\}.$$

Think about this as a function of θ



What happens if we fix θ? We get a line through the origin, and through two points on the unit circle



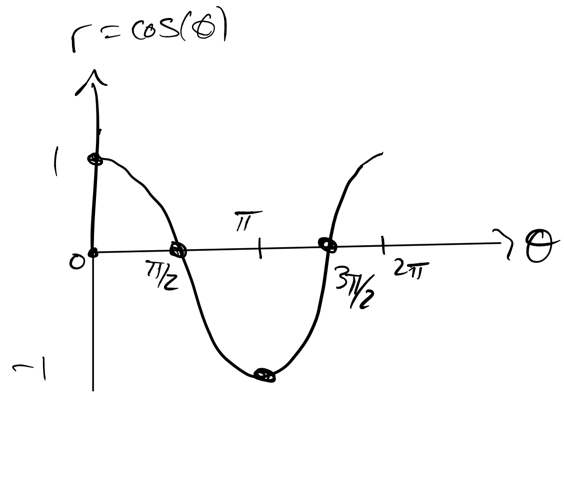
Some interesting graphs

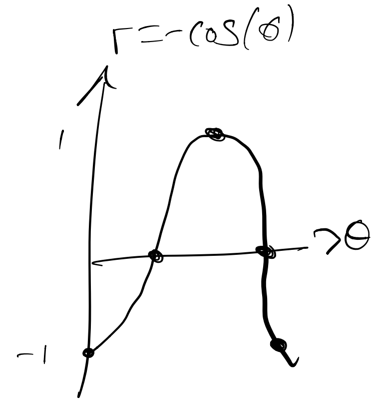
E.g: $r = 1 - \cos(\theta)$

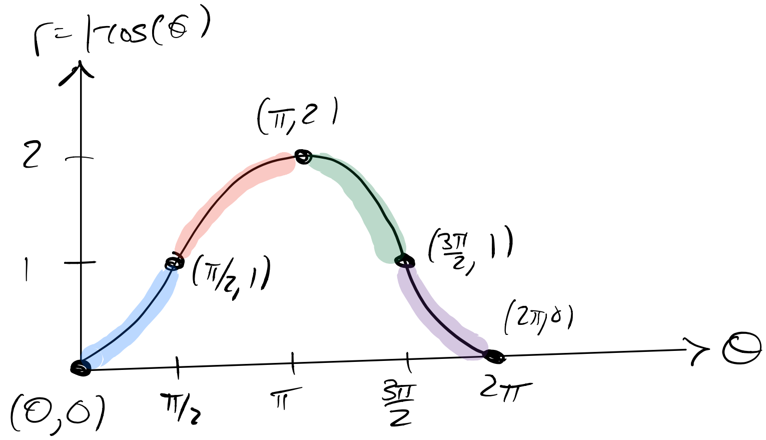
Graph in the Cartesion plane.

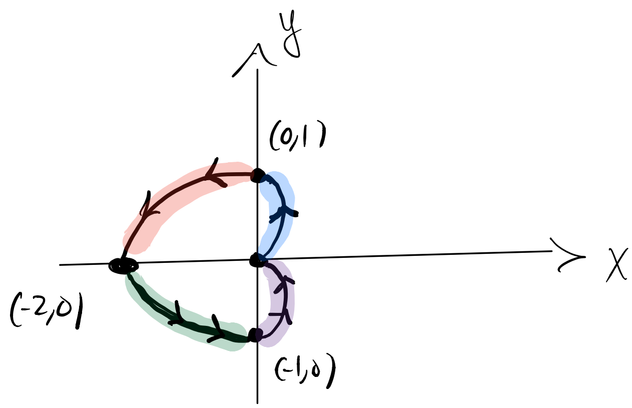
Start by looking at the graph of

$r = 1 - \cos(\theta)$.





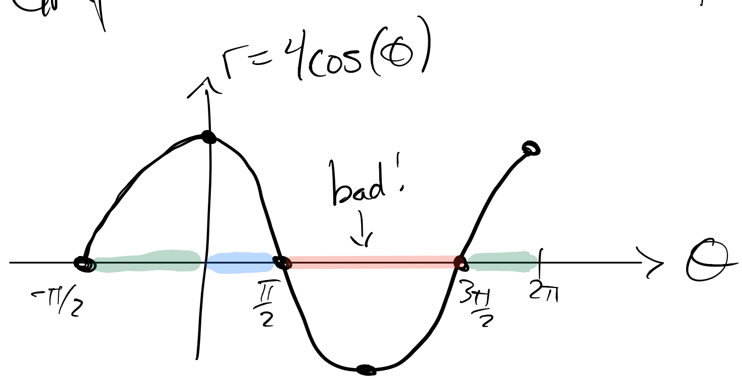




"Cardioid"

E.g: $r^2 = 4\cos(\theta)$

Graph on the Cartesian plane



r is a real number, so $r^2 < 0$ has no solutions.

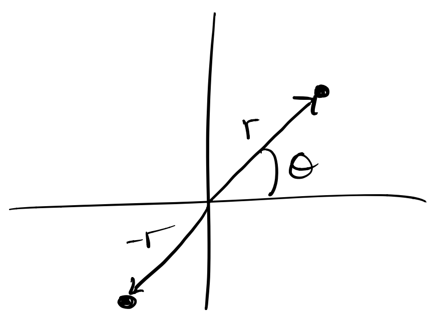
Instead of looking at θ in $[0, \pi/2] \cup [3\pi/2, 2\pi]$, look at θ in $[-\pi/2, \pi/2]$

For a given value of θ, r has two choices:

$$r = \sqrt{4\cos(\theta)} = 2\sqrt{\cos(\theta)} \quad \text{or}$$

$$r = -\sqrt{4\cos(\theta)} = -2\sqrt{\cos(\theta)}.$$

This says we have



Finish next time. Will also do review.