$$f(x) = \frac{1}{(1-x)^3} = (1-x)^{-3}, \quad \frac{d}{dx}(1-x) = -1 \qquad 4/19$$

$$f'(x) = 3(1-x)^{-4} = \frac{(1+2)!}{2}(1-x)^{-(3+1)}$$

$$f''(x) = 12(1-x)^{-5} = 4(3)(1-x)^{-5} = \frac{(2+2)!}{2}(1-x)^{-(3+2)}$$

$$f'''(x) = 60(1-x)^{-6} = 5(4)(3)(1-x)^{-6} = \frac{(3+2)!}{2}(1-x)^{-(3+3)}$$

$$f^{IV}(x) = 360(1-x)^{-7} = 6(5)(4)(3)(1-x)^{-7}$$

$$\frac{4!}{2} = \frac{4(3)(2)}{2} = 4(3) \qquad \frac{5!}{2} = (5)(4)(3)$$

$$f^{(n)}(x) = \frac{(n+2)!}{2}(1-x)^{-(3+n)}$$

$$-(3+n) = -3-n$$

$$f^{(n)}(0) = \frac{(n+2)!}{2}(1-0)^{-(3+n)} = \frac{(n+2)!}{2}$$

$$\sum_{n=0}^{\infty} \frac{f^{n}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{\frac{(n+2)!}{2}}{n!} x^n$$

$$= \sum_{n=0}^{\infty} \frac{(n+2)!}{2} \cdot \frac{1}{n!} x^n$$

$$= \sum_{n=0}^{\infty} \frac{(n+2)!}{2(n!)} x^n = \sum_{n=0}^{\infty} \frac{(n+2)(n+1)}{2} x^n$$

Rmks

① Personally, this seems difficult to me.

② No information about

- convergence of the series.
- whether this power series converges to

$$\frac{1}{(1-x)^3}.$$

If you want info about the first, need to do a Ratio Test.

For the second, need to show that

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1} \to 0$$

as $n \to \infty$

$$R_n(x) = \frac{(n+3)(n+2)}{2} \frac{x^{n+1}}{(1-c)^{3+n+1}}$$

Seems difficult.

Anything obtained from Term-by-Term Integration / Differentiation or substitution, don't need to do <u>any</u> of this extra work.

$$F(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \qquad (-1,1)$$

$$F'(x) = \frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} n x^{n-1} \qquad (-1,1)$$

$$F''(x) = \frac{2}{(1-x)^3} = \sum_{n=0}^{\infty} n(n-1) x^{n-2} \qquad (-1,1)$$

$$\frac{F''(x)}{2} = \frac{1}{(1-x)^3} = \sum_{n=0}^{\infty} \frac{n(n-1)}{2} x^{n-2} \qquad (-1,1)$$

$x\ln(1+2x)$; $f(x)=2x$, $\ln(1+2x)=g(f(x))$

$$g(x)=\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n$$

$$\ln(1+2x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (2x)^n$$

If we think about the Maclaurin Series for the function $h(x)=x$

$$h(0)=0, \ h'(x)=1, \ h'(0)=1$$

$$h''(x)=0=h^{(n)}(x) \text{ for } n\geq 2$$

Get a power series

$$x=\sum_{n=0}^{\infty} a_n x^n, \quad a_n=\frac{h^{(n)}(0)}{n!}=\begin{cases} 0 & \text{if } n\neq 1 \\ 1 & \text{if } n=1 \end{cases}$$

For any $\sum b_n x^n$

$$x\sum b_n x^n = \left(\sum a_n x^n\right)\left(\sum b_n x^n\right) = \sum c_n x^n = \sum b_{n-1} x^n$$

where $c_n = a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0 = b_{n-1}$

If we reindex

$$\sum b_{n-1} x^n = \sum b_n x^{n+1}$$

$$x \ln(1+2x) = x \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (2x)^n = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (2)^n x^n \cdot x$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} 2^n x^{n+1}, \quad -\frac{1}{2} < x \le \frac{1}{2}$$

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n \qquad (-1, 1] \qquad -1 < x \le 1$$

$$\ln(1+2x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} 2^n x^n \qquad -1 < 2x \le 1$$

$$-\frac{1}{2} < x \le \frac{1}{2}$$

Recall $\sum a_n, \sum b_n$ $\quad a_n, b_n$ numbers

If $\sum a_n = A, \sum b_n = B$, then

$$A \pm B = \sum a_n \pm \sum b_n = \sum (a_n \pm b_n)$$

$$f(x), g(x), (f+g)(x) = f(x) + g(x)$$

Power Series

$$f(x) = \sum a_n x^n, \quad \overset{g(x)}{\overset{\parallel}{\sum}} b_n x^n \quad \text{converge for}$$
$$-R < x < R$$

For any value $-R < t < R$

$$f(t) \pm g(t) = \sum a_n t^n \pm \sum b_n t^n$$
$$= \sum (a_n t^n \pm b_n t^n)$$
$$= \sum (a_n \pm b_n) t^n$$

$$\sum a_n x^n \pm \sum b_n x^n = \sum (a_n \pm b_n) x^n$$
for all $-R < x < R$.

$$\ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \underset{\ln(1+(-x))}{\underset{\parallel}{\ln(1-x)}}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (-x)^n = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(-1)^n}{n} x^n = \sum_{n=1}^{\infty} \frac{(-1)^{2n-1}}{n} x^n$$

$$= \sum_{n=1}^{\infty} \frac{-1}{n} x^n = -\sum_{n=1}^{\infty} \frac{1}{n} x^n$$

$$\|$$

<u>other way</u>

$$\int \frac{1}{1-x}\,dx = -\ln(1-x) = -\sum_{n=0}^{\infty} \int x^n\,dx = -\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}$$

$$\ln(1+x) - \ln(1-x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n - \left(-\sum_{n=1}^{\infty} \frac{1}{n} x^n\right)$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n + \sum_{n=1}^{\infty} \frac{1}{n} x^n$$

$$\left(\frac{(-1)^{n-1}+1}{n} x^n\right) = \left(x - \cancel{\tfrac{1}{2}x^2} + \tfrac{1}{3}x^3 - \cancel{\tfrac{1}{4}x^4} + \cdots\right)$$

$$+ \left(x + \cancel{\tfrac{1}{2}x^2} + \tfrac{1}{3}x^3 + \cancel{\tfrac{1}{4}x^4} + \cdots\right)$$

$$= \frac{x^1}{1} + \frac{2}{3}x^3 + \frac{2}{5}x^5 + \frac{2}{7}x^7 + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{2}{2n+1} x^{2n+1}$$