4/3/18

Left off

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(2)}{k!}(x-2)^k = \frac{1}{x} \quad \text{on } (0,4)$$

Def$^{\underline{n}}$: Let f be a function with derivatives of order $k = 1, 2, \ldots, N$, on some interval containing a as an interior point. Then for any integer $0 \le n \le N$, the Taylor polynomial of order n generated by f at $x=a$ is

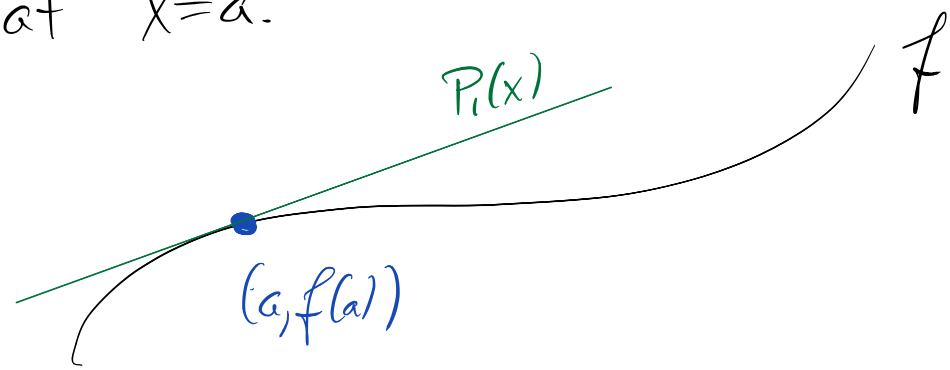
$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!} + \cdots + \frac{f^{(n)}(a)(x-a)^n}{n!}.$$

These are polynomial approximations of degree at most n to the function f on this interval.

Recall from Calc I that

$$P_1(x) = f(a) + f'(a)(x-a)$$

is what the book calls the Linearization of f at $x=a$.



i.e. $P_1(x)$ is the tangent line to f at $x=a$.

<u>E.g.</u>: Find the Taylor polynomials generated by $f(x) = e^x$ at $x=0$.

$$\frac{d}{dx} e^x = e^x, \text{ so } f^{(n)}(0) = e^0 = 1.$$

$$P_n(x) = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \dots + \frac{1}{n!}x^n.$$

E.g: Same for $f(x) = \cos(x)$

$$f(x) = \cos(x) \quad f^{(4k)}(x) = \cos(x) \quad f^{(4k)}(0) = 1$$

$$f'(x) = -\sin(x) \quad f^{(4k+1)}(x) = -\sin(x) \quad f^{(4k+1)}(0) = 0$$

$$f''(x) = -\cos(x) \quad f^{(4k+2)}(x) = -\cos(x) \quad f^{(4k+2)}(0) = -1$$

$$f'''(x) = \sin(x) \quad f^{(4k+3)}(x) = \sin(x) \quad f^{(4k+3)}(0) = 0$$

$$P_n(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \cdots \pm \frac{x^n}{n!}$$

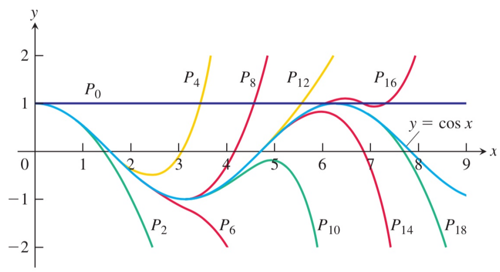
sign depends on whether n is divisible by 4 or not.
+1 if n div by 4
-1 else

We can express the Taylor Series easily, however.

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k}$$

We have nicer way to express the Taylor Polynomial

$$P_n(x) = \sum_{k=0}^{n} \frac{(-1)^k}{(2k)!} x^{2k} = 1 - \frac{x^2}{2} + \frac{x^4}{4!} + \cdots + \frac{(-1)^n}{(2n)!} x^{2n}$$



This is a graph of the first 18 Taylor Polys with cos(x). Note: only has the right side of the graph, because of symmetry.

In particular, $\cos(-x) = \cos(x)$, so $\cos(x)$ is an even function. Any even function is symmetric about the y-axis. Geometrically, this explains why these functions, $P_n(x)$ have no odd terms: if they did, $(-x)^{2k+1} = -(x^{2k+1}) \neq x^{2k+1}$, which would break symmetry.

$$P_n(-x) = \sum_{k=0}^{n} \frac{(-1)^k (-x)^{2k}}{(2k)!} = \sum_{k=0}^{n} \frac{(-1)^k x^{2k}}{(2k)!} = P_n(x).$$

E.g.: It can be shown that

$$f(x) = \begin{cases} 0 & x = 0 \\ e^{\frac{-1}{x^2}} & x \neq 0 \end{cases}$$

has derivatives of all orders, but

$$\sum_{k=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = 0$$

This Taylor Series converges $(-\infty, \infty)$ to 0, but is not equal to the original function on any open interval.

10.9

Thm (Taylor's Theorem)

If f and its first n derivatives,

$$f, f', f'', \ldots, f^{(n)}$$

are continuous on the closed interval $[a,b]$ and

$f^{(n)}$ is differentiable on (a,b), then there is some $a<c<b$ such that

$$(*)\quad f(b) = f(a) + f'(a)(b-a) + \frac{f''(a)}{2!}(b-a)^2 + \dots + \frac{f^{n}(a)}{n!}(b-a)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}.$$

<u>Translation</u>: Taylor poly at $x=a$ of order n is

$$P_n(x) = f(a) + f'(a)(x-a) + \dots + \frac{f^{n}(a)}{n!}(x-a)^n$$

So the equation $(*)$ is

$$f(b) = P_n(a) + \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}$$

So the error from approximating $f(b)$ by $P_n(b)$ is

$$f(b) - P_n(a) = \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}.$$

Taylor's Formula

If f has derivatives of all orders on an open interval, I, containing a as an interior point, then for each positive integer n and each point x in I

$$f(x) = f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x)$$

where

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

for some $a < c < x$.

We can think of this as

$$f(x) = P_n(x) + R_n(x)$$

or

$$f(x) - P_n(x) = R_n(x)$$

We call $R_n(x)$ the Remainder of order n. This is the error incurred by estimating $f(x)$ by $P_n(x)$.

If $R_n(x) \to 0$ as $n \to \infty$ for all $x \in I$, we say the Taylor series generated by f at $x=a$ converges to f on I, and we write

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

Eg: Show

$$f(x) = e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \quad \text{on } \mathbb{R} = (-\infty, \infty).$$

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1} = \frac{e^c}{(n+1)!} x^{n+1}$$

for some c between 0 and x.

Three cases

$x=0$ (§10.1, Thm 5, #6)

$\Rightarrow c=0$

$$R_n(x) = \frac{e^0}{(n+1)!} x^{n+1} = \frac{x^{n+1}}{(n+1)!} \to 0$$

$\underline{x<0}$

$x<c<0 \Rightarrow e^c < e^0 = 1$

$$0 \le |R_n(x)| = \left|\frac{e^c}{(n+1)!}x^{n+1}\right| < \frac{|x|^{n+1}}{(n+1)!} \to 0$$

$\Rightarrow R_n(x) \to 0$

$\underline{x>0}$

$0<c<x \Rightarrow e^c < e^x$

$$0 \le |R_n(x)| = \frac{e^c}{(n+1)!}x^{n+1} < \frac{e^x\, x^{n+1}}{(n+1)!} \to 0$$

$\Rightarrow R_n(x) \to 0.$

$$f(x) = e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \quad \text{on } (-\infty,\infty) = \mathbb{R}.$$