4/12/18

Exam 3 : 4/17/18 (Tuesday)

Power Series & Taylor Series

10.7 – 10.9

2 Bonus problems, one from each past exams. Worth some % of points missed

Good study guide: problems in the back of the sections.

people.math.sc.edu/kustin

Things to know:

Radius of convergence, how to find

- How to use this to determine where a power series converges

i.e. $(a-R, a+R)$, how to check the endpoints.

"Taylor series generated by f at $x=a$"

- What this means, how to compute.

Same for Maclaurin series (i.e. Taylor, $x=0$).

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

Term by Term integration, differentiation.

- where the resulting series converges relative to the original!

Product of Power Series & where the product converges

$$\sum a_n x^n, \quad \sum b_n x^n$$

$$\left(\sum a_n x^n\right)\left(\sum b_n x^n\right) = \sum c_n x^n$$

$$c_n = \sum_{k=0}^{n} a_k b_{n-k}$$

$$= a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \dots + a_n b_0.$$

Substitution

$\sum a_n x^n$, f is a continuous function

$\sum a_n x^n$ converges on I, $f(x) \in I$

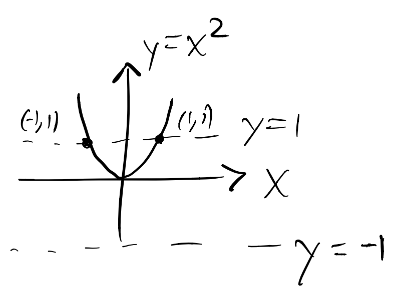
then

$$\sum a_n (f(x))^n \text{ converges.}$$

Eg: Find a power series expansion for $\frac{1}{1-x^2}$ and find the interval of convergence.

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \quad \text{on } (-1,1)$$

$$f(x) = x^2$$



$$-1 < x < 1$$

$$\Rightarrow -1 < f(x) < 1$$

$$\sum_{n=0}^{\infty} (x^2)^n = \frac{1}{1-x^2} \quad \text{on } (-1,1).$$

Taylor's Theorem, Taylor's Remainder Formula

Taylor's Formula: If f has derivatives of all orders on an open interval, I, containing a then for each positive integer n and each point x in I, if $P_n(x)$ is the Taylor Polynomial generated by f at a, then

$$f(x) = P_n(x) + R_n(x)$$

where

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

for some c between x and a.

Should also have some feeling for bounding $f^{(n+1)}(t)$ for t between x and a.

The Taylor Series converges to f if $R_n(x) \to 0$ as $n \to \infty$.

Know these Taylor Series:

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots = \sum_{n=0}^{\infty} x^n, \qquad |x| < 1$$

① $$\frac{1}{1+x} = 1 - x + x^2 - \cdots + (-x)^n + \cdots = \sum_{n=0}^{\infty} (-1)^n x^n, \qquad |x| < 1$$

② $$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \qquad |x| < \infty$$

③ $$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}, \qquad |x| < \infty$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}, \qquad |x| < \infty$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1} \frac{x^n}{n} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}, \qquad -1 < x \le 1$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + (-1)^n \frac{x^{2n+1}}{2n+1} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}, \qquad |x| \le 1$$

maybe not this one.

Remark If you know ①, Term-by-Term integration get $\ln(1+x)$:

$$\ln(1+x) \text{ "=" } \int \frac{(-1)^n}{1+x}\,dx$$

$$\int \frac{1}{1+x}\,dx = \int \sum_{n=0}^{\infty} (-1)^n x^n\,dx \overset{\text{TbT Int}}{=} \sum_{n=0}^{\infty} \int (-1)^n x^n\,dx$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1} + C$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n} + C = \ln(1+x)$$

Check $x=0$

$$\sum_{n=1}^{\infty} \frac{0^n}{n} = \sum_{n=1}^{\infty} 0 = 0.$$

$$\ln(1+0) = \ln(1) = 0$$

$$\Rightarrow \quad C = 0.$$

$$\frac{1}{1-x} = \sum x^n \qquad \frac{1}{1-(-x)} = \frac{1}{1+x} = \sum (-x)^n$$

$$= \sum (-1)^n x^n.$$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}.$$

Know $\quad \sin(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$

Same idea:

$$\frac{d}{dx} \sin(x) = \cos(x),$$

use term by term differentiation.

These techniques are most often associated with problems that look like

"Identify the given power series as a function you know."

10.7 #9

$$\sum_{n=1}^{\infty} \frac{x^n}{n\sqrt{n}\,3^n}$$ What is R, int. of abs. conv., int of conv?

Try Ratio Test

$$\lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)\sqrt{n+1}\,3^{n+1}} \cdot \frac{n\sqrt{n}\,3^n}{x^n} \right|$$

$$=$$

$$\lim_{n\to\infty} \frac{|x|^{n+1}}{|x|^n} \frac{3^n}{3^{n+1}} \frac{n}{n+1} \sqrt{\frac{n}{n+1}}$$

$$=$$

$$\lim_{n\to\infty} |x| \frac{1}{3} \left(\frac{n}{n+1}\right) \sqrt{\frac{n}{n+1}}$$

$\frac{n}{n+1} \to 1$, so $\sqrt{x}$ is continuous

so $\lim_{n\to\infty} \sqrt{\frac{n}{n+1}} = \sqrt{\lim_{n\to\infty} \frac{n}{n+1}} = \sqrt{1} = 1.$

$\lim a_n = A, \ \lim b_n = B \Rightarrow \lim (a_n b_n) = AB$

$$\Rightarrow \lim_{n\to\infty} \frac{|x|}{3}\left(\frac{n}{n+1}\right)\sqrt{\frac{n}{n+1}} = \frac{|x|}{3}(1)(1) = \frac{|x|}{3}$$

Ratio Test says

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1 \Rightarrow \sum a_n \text{ conv. abs.}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| > 1 \Rightarrow \sum a_n \text{ div}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1 \quad \text{inconclusive.}$$

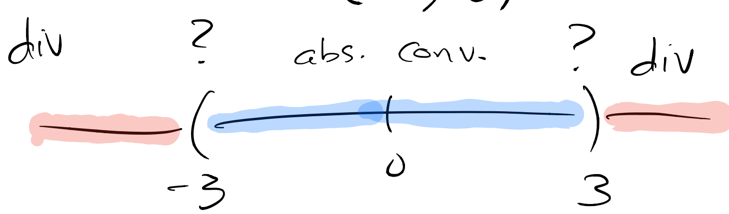
$$\sum_{n=0}^{\infty} \frac{x^n}{n\sqrt{n}\,3^n} \quad \text{conv. abs. when} \quad \frac{|x|}{3} < 1$$

equiv. $\underbrace{|x|}< 3 = R$

equiv.

$-3 < x < 3$

$(-3, 3)$



$\underline{x=3}$

$$\sum_{n=0}^{\infty} \frac{(3)^n}{n\sqrt{n}\ 3^n} = \sum_{n=0}^{\infty} \frac{1}{n\sqrt{n}} = \sum_{n=0}^{\infty} \frac{1}{n^{3/2}}$$

converges $(p = 3/2 > 1)$

$\underline{x=-3}$

$$\sum_{n=0}^{\infty} \frac{(-3)^n}{n\sqrt{n}\ 3^n} = \sum_{n=0}^{\infty} \frac{(-1)^n 3^n}{n\sqrt{n}\ 3^n}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n^{3/2}}$$

converges by the A.C.T

$\sum_{n=1}^{\infty} \frac{x^n}{n\sqrt{n}3^n}$ Converges on $[-3, 3]$.
absolutely.

$\sum_{n=0}^{\infty} (-1)^n u_n$ converges conditionally if

$\sum_{n=0}^{\infty} (-1)^n u_n$ converges but $\sum_{n=0} u_n$ diverges.

$\sum_{n=0}^{\infty} (-1)^n u_n$ converges absolutely if

$$\sum_{n=0}^{\infty} u_n \quad \text{converges, and}$$

$$\text{A.C.T} \Rightarrow \sum_{n=0}^{\infty} (-1)^n u_n \quad \text{also converges.}$$