4/10/18

Estimating the Remainder

Thm (The Remainder Estimation Theorem):

If there is a positive constant, M, such that

$$|f^{(n+1)}(t)| \le M$$

for all t between x and a, then

$$|R_n(x)| \le M \frac{|x-a|^{n+1}}{(n+1)!}.$$

If this condition holds for every n and the other conditions of Taylor's Theorem are satisfied, then the Taylor series converges to f.

Recall Taylor's Theorem states that if f, $f', f'', \dots, f^{(n)}$ are all continuous on $[a,b]$, $f^{(n)}$ is differentiable on (a,b), then there exists some $a < c < b$ such that

$$f(b) = P_n(b) + \underbrace{\frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}}_{R_n}$$

The condition that $\left|f^{(n+1)}(t)\right| \le M_n$ for all $a<t<b$ for all n says that: for any choice of n

$$|R_n(x)| \le \frac{M_n\,|x-a|^{n+1}}{(n+1)!}$$

E.g: Show that the Taylor Series for $\sin(x)$ at $x=0$ converges to sine everywhere.

$$f(x) = \sin(x) \qquad f^{(4k)}(x) = \sin(x)$$

$$f'(x) = \cos(x) \qquad f^{(4k+1)}(x) = \cos(x)$$

$$f''(x) = -\sin(x) \qquad f^{(4k+2)}(x) = -\sin(x)$$

$$f'''(x) = -\cos(x) \qquad f^{(4k+3)}(x) = -\cos(x)$$

$$f^{(4k)}(0) = 0, \quad f^{(4k+1)}(0) = 1, \quad f^{(4k+2)}(0) = 0$$

$$f^{(4k+3)}(0) = -1.$$

$$0 + x + \frac{0}{2!}x^2 + \frac{(-1)x^3}{3!} + \frac{0x^4}{4!} + \frac{1x^5}{5!} + \cdots$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \cdots$$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$$

The only derivatives that occur are $\pm\sin(x)$ and $\pm\cos(x)$, both are bounded in absolute value by 1:

$$|\pm\sin(x)| = |\sin(x)| \le 1, \quad |\pm\cos(x)| = |\cos(x)| \le 1$$

$$\Updownarrow \qquad\qquad\qquad \Updownarrow$$

$$-1 \le \sin(x) \le 1 \qquad\qquad -1 \le \cos(x) \le 1$$

So for any n,

$$\left| f^{(n+1)}(t) \right| \le 1$$

for any x and any t between 0 and x.

$$|R_n(x)| \le \frac{|x|^{2k+2}}{(2k+2)!}, \qquad n = 2k+1$$

So $R_n(x) \to 0$ by the Theorem from 10.1 that says

$$\lim_{n\to\infty} \frac{x^n}{n!} = 0, \text{ for all } x.$$

This says

$$\sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$$

A similar argument says

$$\cos(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} \quad \text{for all } x.$$

E.g: Find the first few terms of

$$\frac{1}{3}(2x + x\cos(x))$$

$$\frac{1}{3}\left(2x + x\left(1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \cdots\right)\right)$$

$$= \frac{1}{3}\left(2x + x - \frac{x^3}{2} + \frac{x^5}{4!} - \frac{x^7}{6!} + \frac{x^9}{8!} - \frac{x^{11}}{10!} + \cdots\right)$$

$$= \frac{1}{3}\left(3x - \frac{x^3}{2} + \frac{x^5}{4!} - \frac{x^7}{6!} + \frac{x^9}{8!} - \frac{x^{11}}{10!} + \cdots\right)$$

$$= x - \frac{x^3}{6} + \frac{x^5}{3\cdot 4!} - \frac{x^7}{3\cdot 6!} + \frac{x^9}{3\cdot 8!} - \frac{x^{11}}{3\cdot 10!} + \cdots$$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{3(2k)!}$$

E.g: Find the first few terms of $e^x \cos(x)$.

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}, \quad \cos(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$$

Recall: $\sum a_n x^n$, $\sum b_n x^n$ convergent

$$\sum c_n x^n = \sum a_n x^n \sum b_n x^n$$

$$c_n = a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \dots + a_n b_0$$

converges.

Here a_n = coefficients for e^x, b_n = coefficients for $\cos(x)$.

Want to know the first few c_n. Note the b_n's with odd index are all zero.

$$c_0 = a_0 b_0 = (1)(1) = 1$$

$$c_1 = a_0 b_1 + a_1 b_0 = a_1 b_0 = (1)(1) = 1$$

$$c_2 = a_0 b_2 + \cancel{a_1 b_1}^{=0} + a_2 b_0 = a_0 b_2 + a_2 b_0$$

$$= (1)\frac{(-1)^1}{(2(1))!} + \left(\frac{1}{2!}\right)(1) = \frac{-1}{2} + \frac{1}{2} = 0.$$

$$C_3 = a_0 \cancel{b_3}^{\to 0} + a_1 b_2 + \cancel{a_2 b_1}^{\to 0} + a_3 b_0$$

$$a_n = \frac{1}{n!}$$

$$b_{2k} = \frac{(-1)^k}{(2k)!}$$

$$= a_1 b_2 + a_3 b_0$$

$$= (1)\left(\frac{-1}{2}\right) + \frac{1}{3!}(1)$$

$$= \frac{-1}{2} + \frac{1}{6} = \frac{-3}{6} + \frac{1}{6} = \frac{-2}{6} = \frac{-1}{3}.$$

$$C_4 = a_0 b_4 + \cancel{a_1 b_3}^{\to 0} + a_2 b_2 + \cancel{a_3 b_1}^{\to 0} + a_4 b_0$$

$$= a_0 b_4 + a_2 b_2 + a_4 b_0$$

$$= 1\left(\frac{(-1)^2}{4!}\right) + \left(\frac{1}{2}\right)\left(\frac{(-1)^1}{(2(1))!}\right) + \frac{1}{4!}(1)$$

$$= \frac{1}{4!} - \frac{1}{4} + \frac{1}{4!}$$

$$= \frac{2}{4!} - \frac{1}{4} = \frac{1}{4 \cdot 3} - \frac{1}{4} = \frac{1}{12} - \frac{3}{12} = \frac{-2}{12}$$

$$= \frac{-1}{6}.$$

$$e^x \cos(x) = 1 + x - \frac{1}{3}x^3 - \frac{1}{6}x^4 + \dots$$

Eg.: Find a Taylor Series for $\cos(2x)$.

$$\cos(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} \quad \text{on } (-\infty, \infty)$$

$2x$ is a continuous function on $(-\infty,\infty)$ with range $(-\infty, \infty)$

$$\overset{\text{Theorem 20}}{\Longrightarrow} \cos(2x) = \sum_{k=0}^{\infty} \frac{(-1)^k (2x)^{2k}}{(2k)!}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k 4^k x^{2k}}{(2k)!} \quad \text{on } (-\infty,\infty).$$

E.g: (Small Angle Approximation)

For "small" θ, $\sin(\theta) \approx \theta$. The justification is an application of Taylor's Remainder Theorem.

$$\sin(\theta) = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots$$

$$\sin(\theta) = P_1(\theta) + R_1(\theta) = \theta + R_1(\theta)$$

$$|R_1(\theta)| \le \frac{\theta^2}{2!} = \frac{\theta^2}{2}$$

Say, e.g., $\theta = \frac{1}{10}$, then $|R_1(\theta)| \leq \frac{1}{200}$.

Exam 3 next Tuesday, covers Power Series & Taylor Series