3/29/18

<u>Thm</u> (The Term-by Term Differentiation):

If $\sum' c_n(x-a)^n$ has radius of convergence $R>0$, it defines a function

$$f(x) = \sum' c_n(x-a)^n \quad \text{on } (a-R, a+R)$$

with derivatives of all orders on this interval and the derivatives are obtained by differentiating term-by-term

$$f'(x) = \sum' \frac{d}{dx}\left(c_n(x-a)^n\right)$$

$$= \sum' c_n \frac{d}{dx}(x-a)^n$$

$$= \sum' c_n n (x-a)^{n-1}$$

$$f''(x) = \frac{d}{dx} f'(x) = \sum' \frac{d}{dx} c_n n (x-a)^{n-1}$$

$$= \sum' c_n (n)(n-1)(x-a)^{n-2}$$

$\vdots$

$$f^{(k)}(x) = \sum c_n (n)(n-1)(n-2)\cdots\underbrace{(n-(k-1))}_{(n-k+1)}(x-a)^{n-k}$$

$$= \sum \frac{n!}{(n-k)!} c_n (x-a)^{n-k}$$

Each of these series converge on $(a-R, a+R)$

E.g: Find the first two derivatives of $\frac{1}{1-x}$.

$$\frac{d}{dx}(1-x)^{-1} = -1(1-x)^{-2}(-1) = \frac{1}{(1-x)^2}$$

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$$\frac{d^2}{dx^2}((1-x)^{-1}) = \frac{d}{dx}((1-x)^{-2}) = -2(1-x)^{-3}(-1) = \frac{2}{(1-x)^3}$$

Know $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad (-1,1)$

$$\frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{d}{dx}\sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty}\frac{d}{dx}x^n$$

$$= \frac{d}{dx}(1) + \frac{d}{dx}(x) + \frac{d}{dx}(x^2) + \frac{d}{dx}(x^3) + \cdots$$

$$= 0 + 1 + 2x + 3x^2 + \cdots$$

$$= \sum_{n=1}^{\infty} n x^{n-1} = \frac{1}{(1-x)^2}$$

$$\frac{d^2}{dx^2}\left(\frac{1}{1-x}\right) = \frac{2}{(1-x)^3} = \sum_{n=1}^{\infty} \frac{d}{dx}\left(n x^{n-1}\right)$$

$$= \sum_{n=1}^{\infty} n(n-1) x^{n-2}$$

$$= \sum_{n=2}^{\infty} n(n-1) x^{n-2} \qquad \text{on } (-1,1)$$

<u>Thm</u> (The Term-by Term Integration Theorem)

Suppose that

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

Converges on $(a-R, a+R)$, $R>0$. Then

$$\sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1} = \sum_{n=0}^{\infty} c_n \left(\int (x-a)^n dx \right)$$

an anti-deriv not family of functions $C=0$

Converges on $(a-R, a+R)$ and

$$\int f(x)\,dx = \left(\sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1} \right) + C.$$

on $(a-R, a+R)$.

Eg: Identify the function

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} \quad \text{on } (-1, 1)$$

Observe that

$$f'(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (2n+1) x^{2n}}{2n+1}$$

$$= \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

Recall

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$= \sum_{n=0}^{\infty} (-1)^n (x^2)^n$$

$$= \sum_{n=0}^{\infty} (-x^2)^n = \frac{1}{1-(-x^2)} = \frac{1}{1+x^2}$$

on $(-1, 1)$ $\quad |-x^2| < 1 \iff |x|^2 < 1 \iff |x| < 1$

This says that

$$f'(x) = \frac{1}{1+x^2}$$

so by Term-By-Term Integration & the FTC

$$f(x) = \int f'(x)\,dx = \int \frac{dx}{1+x^2} = \arctan(x) + c$$

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = \arctan(x) + c.$$

$$f(0) = \sum_{n=0}^{\infty} \frac{(-1)^n 0^{2n+1}}{2n+1} = 0 = \arctan(0) + c = c$$

So

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = \arctan(x) \quad \text{on } (-1,1).$$

Eg: The series

$$\frac{1}{1+t} = \frac{1}{1-(-t)} = \sum_{n=0}^{\infty} (-t)^n \qquad -1 < t < 1$$

$$\int \frac{dt}{1+t} = \ln|1+t| + C$$

$$\int \sum_{n=0}^{\infty} (-t)^n \, dt = \sum_{n=0}^{\infty} \frac{(-t)^{n+1}(-1)}{n+1}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(-1)\, t^{n+1}}{n+1}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n t^{n+1}}{n+1}$$

$$= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{t^n}{n}$$

So on $(-1,1)$

$$\ln|1+t| = \left(\sum_{n=1}^{\infty} (-1)^{n-1} \frac{t^n}{n}\right) + C$$

$$\ln(1) = 0 = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{0^n}{n} + C = 0 + C$$

$\Rightarrow C = 0$, $\ln|1+t| = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{t^n}{n}$ on $(-1,1)$.

Can show that this holds on $(-1,1]$,

as a corollary

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{(1)}{n} = \ln(2).$$

↑ Alternating Harmonic Series.

10.8 Taylor and Maclaurin Series

Def^{n} Let f be a function with derivatives of all orders on an interval containing a as an interior point. The Taylor series generated by f at $x=a$ is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

The Maclaurin series of f is the Taylor series generated by f at $x=0$.

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = f(0) + f'(0)x + f''(0)x^2 + \cdots$$

E.g: Find the Taylor series of $f(x) = \frac{1}{x}$ at 2.

$k=0$ $f(x) = \frac{1}{x} = x^{-1}$

$k=1$ $f'(x) = -x^{-2}$

$k=2$ $f''(x) = 2x^{-3}$

$k=3$ $f'''(x) = -3(2)x^{-4}$

$f^{(k)}(x) = (-1)^k k!\, x^{-(k+1)}$, $f^{(k)}(2) = (-1)^k k! \frac{1}{2^{k+1}}$

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(2)}{k!}(x-2)^k = \sum_{k=0}^{\infty} \frac{(-1)^k k!}{2^{k+1} k!}(x-2)^k$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{k+1}} (x-2)^k$$

Think of

$$\frac{(-1)^k (x-2)^k}{2^{k+1}} = \frac{(-(x-2))^k}{2\,2^k} = \underbrace{\frac{1}{2}}_{a} \underbrace{\left(\frac{-(x-2)}{2}\right)}_{r}^k$$

Converges when

$$|r| = \left|\frac{-(x-2)}{2}\right| = \frac{|x-2|}{2} < 1$$

$$\Rightarrow \quad |x-2| < 2 \iff -2 < x-2 < 2$$

$$\iff 2-2=0 < x < 2+2=4$$

$$\sum_{k=0}^{\infty} \frac{1}{2}\left(\frac{-(x-2)}{2}\right)^k = \frac{\left(\frac{1}{2}\right)}{1-\left(\frac{-(x-2)}{2}\right)} = \frac{1}{2\left(1+\frac{x-2}{2}\right)}$$

$$= \frac{1}{2+x-2} = \frac{1}{x} \text{ on } (0,4)$$