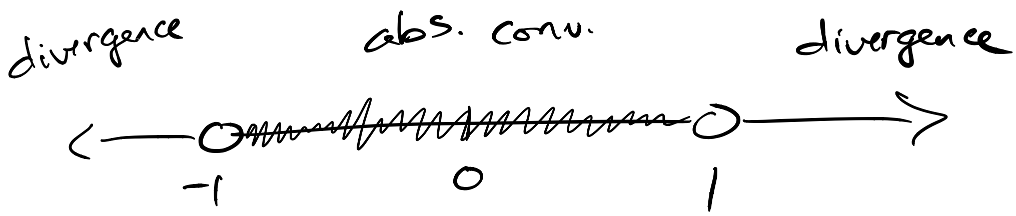
3/27/18

Eg: $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$

(Thurs)

Ratio test, $\lim_{n \to \infty} \frac{|x^{n+1}|}{n+1} \frac{n}{|x^n|} = |x| < 1$



$\underline{x = -1}$

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{(-1)^n}{n}$$

$$\parallel$$

$$\sum_{n=1}^{\infty} \frac{-1}{n}$$ - diverges because it is a constant multiple of the divergent Harmonic Series.

$$(-1)^{n-1}(-1)^n = (-1)^{n+n-1}$$
$$= (-1)^{2n-1}$$
$$= ((-1)^2)^n (-1)^{-1}$$
$$= \frac{1}{-1} = -1.$$

$\underline{x=1}$ $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{(1)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$

Alternating Harmonic Series, converges.

So $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$ converges on $(-1, 1]$.

$\underline{Eg.}$: $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n-1}}{2n-1}$

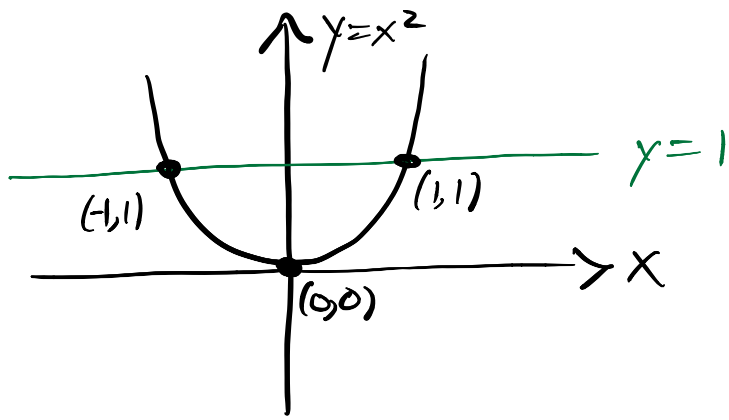
$$\lim_{n\to\infty} \left| \frac{x^{2(n+1)-1}}{2(n+1)-1} \frac{2n-1}{x^{2n-1}} \right| =$$

$$\lim_{n\to\infty} \left| \frac{x^{2n+1}}{x^{2n-1}} \right| \frac{2n-1}{2n+1} = \lim_{n\to\infty} |x^2| \frac{2n-1}{2n+1}$$

$$= |x|^2 = x^2$$

By the Ratio Test, this series converges for $x^2 < 1$, or $|x| < 1$

$-1 < x < 1$



Check endpoints: $x = -1$

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{(-1)^{2n-1}}{2n-1} = \sum_{n=1}^{\infty} \frac{(-1)^{3n-2}}{2n-1}$$

$$(-1)^{3n} = \left((-1)^3\right)^n = (-1)^n, \quad (-1)^{-2} = \frac{1}{(-1)^2} = 1.$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{3n-2}}{2n-1} = \sum_{n=1}^{\infty} \frac{(-1)^n}{2n-1} =$$

$$-\frac{1}{1} + \frac{1}{3} - \frac{1}{5} + \cdots$$

$$u_n = \frac{1}{2n-1}, \quad 1, \ 1/3, \ 1/5, \ 1/7, \ \ldots$$

$$0 < u_n, \quad u_{n+1} = \frac{1}{2n+1} < \frac{1}{2n-1} = u_n, \quad u_n \to 0$$

So this series converges by the A.S.T.

$\underline{x=1}$

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{2n-1}$$

$u_n = \frac{1}{2n-1}$ converges by the A.S.T. just as in the last case.

So the power series

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n-1}}{2n-1} \text{ converges on } [-1, 1]$$

$\underline{E.g}$: $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots$

$$\lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| = \lim_{n\to\infty} \frac{|x|}{n+1} = 0 < 1$$

So the power series converges for every real number x: $(-\infty, \infty) = \mathbb{R}$

Eg: $\sum_{n=0}^{\infty} n!\, x^n = 1 + 2x^2 + 3!\,x^3 + 4!\,x^4 + \cdots$

$$\lim_{n\to\infty} \left| \frac{(n+1)!\, x^{n+1}}{n!\, x^n} \right| = \lim_{n\to\infty} (n+1)|x| = \begin{cases} \infty & x \neq 0, \\ 0 & x = 0 \end{cases}$$

So this power series converges only if $x=0$.

So far we've seen examples of power series that converge

on $(-a, a)$

on $(-a, a]$ or $[-a, a)$

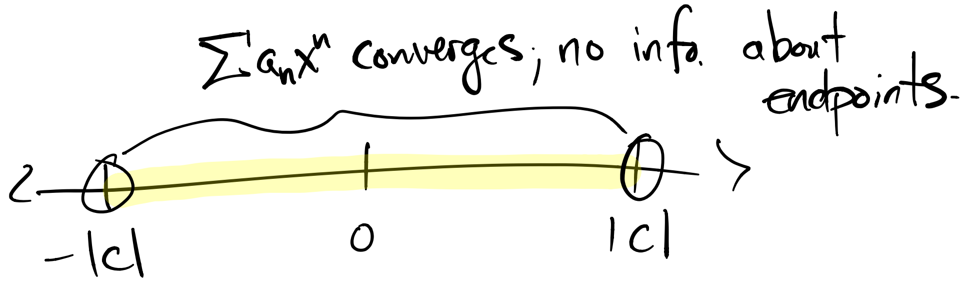
on $(-\infty, \infty) = \mathbb{R}$

at a single point.

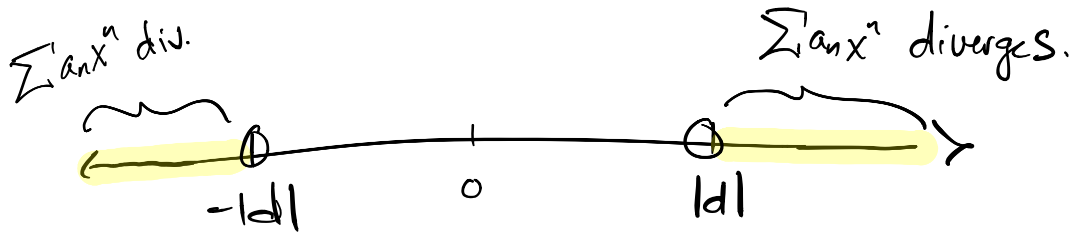
Thm (The Convergence Theorem for Power Series)

If the power series $\sum_{n=0}^{\infty} a_n x^n$ converges at $x = c \neq 0$, then it converges absolutely for all x with $|x| < |c|$.

If the power series diverges at $x = d$, then it diverges for all x with $|d| < |x|$.



$-|c| < x < |x|$ equiv. to $|x| < |c|$.



$|d| < |x|$

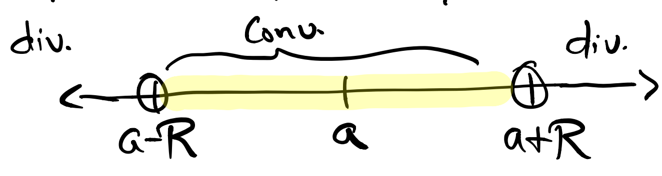
Proof: In the book.

Rmk: This is also true if $\sum a_n x^n$ is replaced by $\sum a_n (x-a)^n$, a some fixed real number.

Radius of Convergence

Corollary: The convergence of the series $\sum_{n=0}^{\infty} c_n (x-a)^n$ is described by one of the following three cases:

1. There is a positive number, R, such that the series diverges for all x with $|x-a| > R$, but converges absolutely for all x with $|x-a| < R$. The series may or may not converge at $x = a-R$ and $x = a+R$.



$(a-R, a+R)$

2. The series converges on $\mathbb{R} = (-\infty, \infty)$
$(R = \infty)$

3. The series converges at $x = a$ and diverges everywhere else.
$(R = 0)$.

We call R the Radius of Convergence of the series $\sum_{n=0}^{\infty} c_n (x-a)^n$

How to Test a Power Series for Convergence

1. Use the ratio or root test to find R
This gives the interval
$$|x-a| < R \quad \text{or} \quad a-R < x < a+R$$
on which the series converges absolutely.
2. If R is finite and positive, test the endpoints $x = a-R$ and $x = a+R$.

3. For all x such that $x < a-R$ or $x > a+R$, the series diverges.

<u>Operations</u> <u>on</u> <u>Power</u> <u>Series</u>

<u>Thm</u> (The Series Multiplication Theorem for Power Series)

If $A(x) = \sum_{n=0}^{\infty} a_n x^n$ and $B(x) = \sum_{n=0}^{\infty} b_n x^n$

both converge absolutely on $|x| < R$, and

$$c_n = a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \cdots + a_n b_0$$

then

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots$$

converges absolutely to $A(x)B(x)$ for $|x| < R$:

$$\left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty} c_n x^n.$$

Thm: If $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely for $|x|<R$, and f is a continuous function, then

$$\sum_{n=0}^{\infty} a_n (f(x))^n$$

converges absolutely for $|f(x)|<R$.

E.g: We know

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \qquad |x|<1$$

So

$$\frac{1}{1-4x^2} = \sum_{n=0}^{\infty} (4x^2)^n \qquad |4x^2|<1$$

$$= \sum_{n=0}^{\infty} 4^n x^{2n} \qquad |x|^2 < \frac{1}{4}$$

$$\Updownarrow$$

$$|x| < \frac{1}{2}$$