Hint: Geometric 3/22/18

1. $\sum_{n=0}^{\infty} \frac{5(3^n)+2^{n+1}}{6^n}$ Determine if conv./div. find the sum.

$$\sum_{n=0}^{\infty} \frac{5(3^n)+2^{n+1}}{6^n} = \sum_{n=0}^{\infty}\left(\frac{5(3^n)}{6^n}+\frac{2^{n+1}}{6^n}\right) = \sum_{n=0}^{\infty}\left(5\left(\frac{3}{6}\right)^n+2\left(\frac{2}{6}\right)^n\right)$$

$$= \sum_{n=0}^{\infty}\left(5(1/2)^n+2\left(\tfrac{1}{3}\right)^n\right) = \sum_{n=0}^{\infty} 5\left(\tfrac{1}{2}\right)^n + \sum_{n=0}^{\infty} 2(1/3)^n$$

$$= \frac{5}{1-1/2} + \frac{2}{1-1/3} = \frac{5}{\frac{1}{2}} + \frac{2}{2/3} = 5(2)+2(3/2)=13.$$

2. $\sum_{n=0}^{\infty} \frac{e^n}{e^n+n}$ conv/div?

$$\lim_{n\to\infty} \frac{e^n}{e^n+n} \overset{\text{L'Hôpital}}{=} \lim_{n\to\infty} \frac{e^n}{e^n+1} \overset{\text{L'H}}{=} \lim_{n\to\infty} \frac{e^n}{e^n} = \lim_{n\to\infty} 1 = 1 \neq 0$$

nth Term Test for Divergence says

$\sum_{n=0}^{\infty} \frac{e^n}{e^n+n}$ diverges.

3. $\sum_{n=1}^{\infty} \frac{\ln(n)}{n}$

Easiest: compare to $\sum_{n=1}^{\infty} \frac{1}{n}$ observe $e < \overset{N}{\overset{\shortparallel}{3}} < n$, $\ln(x)$ increasing

$$\ln(e)=1<\ln(3)<\ln(n)$$

$$\Rightarrow \quad \frac{1}{n} < \frac{\ln(n)}{n} \Rightarrow \sum_{n=1}^{\infty} \frac{\ln(n)}{n} \text{ diverges by comp.}$$

Also could use L.C.T. with Harmonic

$$\lim_{n\to\infty} \frac{\ln(n)}{n} \Big/ \frac{1}{n} = \lim_{n\to\infty} \frac{n\ln(n)}{n} = \lim_{n\to\infty} \ln(n) = \infty$$

Part (3) of L.C.T.: $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum b_n$ diverges,
then $\sum a_n$ diverges

Here $a_n = \frac{\ln(n)}{n}$, $b_n = 1/n$, so $\sum_{n=1}^{\infty} \frac{\ln(n)}{n}$ diverges.

4. $\sum_{n=1}^{\infty} \frac{n+1}{n^2\sqrt{n}}$

Could use L.C.T. w/ $\sum_{n=1}^{\infty} 1/n^{3/2}$ (conv. p-series)

Easy: $n^2\sqrt{n} = n^2 n^{1/2} = n^{4/2} n^{1/2} = n^{5/2}$

$$\frac{n+1}{n^2\sqrt{n}} = \frac{n+1}{n^{5/2}} = \frac{n}{n^{5/2}} + \frac{1}{n^{5/2}} = \frac{1}{n^{3/2}} + \frac{1}{n^{5/2}}, \quad 1 < 3/2, 5/2$$

$$\sum_{n=1}^{\infty} \frac{n+1}{n^2\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}} + \sum_{n=1}^{\infty} \frac{1}{n^{5/2}}$$, both conv. p-series

So $\sum_{n=1}^{\infty} \frac{n+1}{n^2\sqrt{n}}$ converges.

5. $\sum_{n=1}^{\infty} (-1)^n \frac{n^2 (n+2)!}{n!\, 3^{2n}}$ conv conditionally, conv. absolutely or div?

Ratio Test

$$\lim_{n\to\infty} \frac{(n+1)^2(n+3)!}{(n+1)!\,3^{2n+2}} \frac{n!\,3^{2n}}{n^2(n+2)!} = \lim_{n\to\infty} \frac{(n+1)^2}{n^2} \frac{(n+3)!}{(n+2)!} \frac{n!}{(n+1)!} \frac{\cancel{3^{2n}}}{\cancel{3^{2n}}} \frac{1}{3^2}$$

$$= \lim_{n\to\infty} \frac{(n+1)^2}{n^2}(n+3)\left(\frac{1}{n+1}\right)\frac{1}{9} = \lim_{n\to\infty} \frac{(n+1)(n+3)}{9n^2}$$

$$= \lim_{n\to\infty} \frac{n^2+4n+3}{9n^2} = \frac{1}{9} < 1$$

Converges absolutely by the Ratio Test.

6. $\sum_{n=1}^{\infty} \left(\frac{4n+3}{3n-5}\right)^n$

Root Test: $n \geq 2$

$$\lim_{n\to\infty} \sqrt[n]{\left|\frac{4n+3}{3n-5}\right|^n} = \lim_{n\to\infty} \frac{4n+3}{3n-5} = \frac{4}{3} > 1$$

Diverges by the Ratio Test.

7. $\sum_{n=1}^{\infty} (-1)^n \frac{2n}{4n^2+1}$ conv. cond., abs., or diverges?

L.C.T. w/ $\sum 1/n$

$$\lim_{n\to\infty} \frac{2n}{4n^2+1} \Big/ \frac{1}{n} = \lim_{n\to\infty} \frac{2n^2}{4n^2+1} = \frac{2}{4} = \frac{1}{2} > 1$$

This is clearly nonsense.
This should be $\frac{1}{2} > 0$.

$$\Rightarrow \sum_{n=1}^{\infty} \frac{2n}{4n^2+1} \text{ diverges.}$$

$$\parallel$$

$$\sum_{n=1}^{\infty} \left|(-1)^n \frac{2n}{4n^2+1}\right|$$

This says the Ratio and Root Tests can never give convergence, only inconclusive or diverges.

$$u_n = \frac{2n}{4n^2+1}$$

1. $0 < u_n$ ✓

2. $\underline{u_{n+1} \leq u_n}$ eventually

3. $\lim\limits_{n\to\infty} u_n = 0$. ✓ $\lim\limits_{n\to\infty} \frac{2n}{4n^2+1} = 0.$

$$f(x) = \frac{2x}{4x^2+1}, \quad f'(x) = \frac{2(4x^2+1) - 2x(8x)}{(4x^2+1)^2} < 0$$

only happens if $2(4x^2+1) - 2x(8x) < 0$

$$8x^2 + 2 - 16x^2 = -8x^2 + 2 < 0$$

$$\iff 2 < 8x^2$$

$$\iff \frac{1}{4} = \frac{2}{8} < x^2$$

$$\iff \frac{1}{\sqrt{4}} = \frac{1}{2} < x \ ✓$$

Conditional convergence by the A.S.T.

8. $\sum\limits_{n=1}^{\infty} \sin(1/n)$

L.C.T. w/ $\sum 1/n$. Let $h = 1/n$ $\lim_{n\to\infty} h = 0$

$$\lim_{n\to\infty} \frac{\sin(1/n)}{\frac{1}{n}} = \lim_{h\to 0} \frac{\sin(h)}{h} = 1 > 0$$

Diverges by L.C.T.

10.7 Power Series

Def$^{\underline{n}}$: A power series about $x=a$ is a series of the form

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

A power series about $x=0$ is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots$$

The constant a is called the center and the constants $c_0, c_1, c_2, \ldots, c_n, \ldots$

are the coefficients.

We can think of these as functions with domain the points where

$$\sum_{n=0}^{\infty} c_n x^n$$

converges.

E.g: Geometric series are basically power series.

$$f(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \quad |x|<1 \text{ or } -1<x<1$$

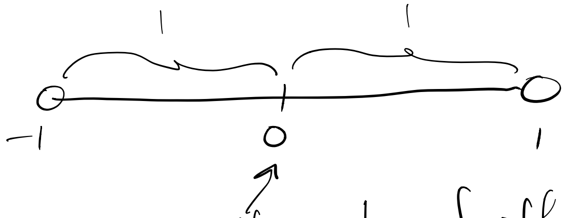
This is a power series expansion for

$$\frac{1}{1-x}.$$

The set of points on the real line for which

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots$$

converges is an interval



in the center of the interval.

Recall that one can estimate the sum of a series

$$\sum_{n=0}^{\infty} a_n = a_0 + a_1 + a_2 + \cdots + a_n + \cdots$$

using the partial sums

$$S_n = a_0 + a_1 + a_2 + \cdots + a_n$$

because

$$\sum_{n=0}^{\infty} a_n := \lim_{n \to \infty} S_n = S$$

By definition this means that for any $\varepsilon > 0$ we can always find some integer N such that

$$|S - S_n| < \varepsilon$$

when $N \leq n$. Equivalently,

$$-\varepsilon < S - S_n < \varepsilon$$

or

$$S_n - \varepsilon < S < S_n + \varepsilon. \quad \text{for } N \le n.$$

In particular

$$S_N - \varepsilon < S < S_N + \varepsilon.$$

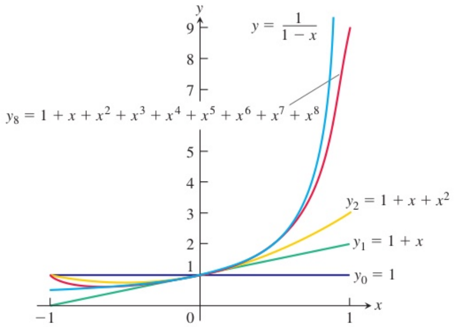
For a power series, $\sum_{n=0}^{\infty} c_n x^n$, the analogue of partial sums is

$$P_n(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n$$

which we can think of as a polynomial approximation to the function

$$f(x) = \sum_{n=0}^{\infty} c_n x^n.$$

Say for $f(x) = \frac{1}{1-x}$, these give polynomial approximations to a function that is <u>not</u> a polynomial.



Eg: Find the values of x for which

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$$

Converges.

Ratio test:

$$\lim_{n\to\infty} \left| \frac{(-1)^n x^{n+1}}{n+1} \cdot \frac{n}{(-1)^{n-1} x^n} \right| = \lim_{n\to\infty} \frac{|x|^{n+1}}{|x|^n} \frac{n}{n+1}$$

$$= \lim_{n\to\infty} |x| \frac{n}{n+1} = \lim_{n\to\infty} |x| \lim_{n\to\infty} \frac{n}{n+1}$$

$$= |x| < 1$$

As long as $|x| < 1$, Ratio Test $\Rightarrow$ abs conv.