3/8/18

E.g: $0 < p < 1$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^p} = 1 - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \cdots$$

$$u_n = \frac{1}{n^p}, \qquad u_{n+1} = \frac{1}{(n+1)^p} \le \frac{1}{n^p} = u_n, \quad u_n \to 0$$

$$0 \le u_n$$

By the A.S.T, this series converges. This is conditionally convergent because

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$ diverges by Integral Test $0 < p < 1$.

Thm (The Rearrangement Theorem for Absolutely Convergent Series)

If $\sum a_n$ converges absolutely and

$$b_1, b_2, b_3, \ldots, b_n, b_{n+1}, \ldots$$

is any rearrangement of $\{a_n\}_{n=1}^{\infty}$, then $\sum b_n$ converges absolutely and

$$\sum b_n = \sum a_n.$$

Caution This is **NOT** true if $\sum a_n$ converges conditionally.

We know $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = L$ for some L.

So

$$2L = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n}$$

$$= 2 - 1 + \tfrac{2}{3} - \tfrac{1}{2} + \tfrac{2}{5} - \tfrac{1}{3} + \tfrac{2}{7} - \tfrac{1}{4} + \cdots$$

$$= (2-1) - \tfrac{1}{2} + \left(\tfrac{2}{3} - \tfrac{1}{3}\right) - \tfrac{1}{4} + \left(\tfrac{2}{5} - \tfrac{1}{5}\right) - \tfrac{1}{6} + \cdots$$

$$= 1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} - \tfrac{1}{6} + \cdots$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = L$$

$\Rightarrow 2L = L \Rightarrow 2L - L = L = 0$

But $L \neq 0$, contradiction. So the rearrangement of the Alternating Harmonic Series does not converge to $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$.

Review (Exam 2, 3/20/18)

- 10.1: Sequences
- 10.2: Series stuff (Geometric), n^{th} term test
- 10.3: Integral Test
- 10.4 Comparison Tests
- 10.5 : Abs. Conv., Ratio & Root Tests
- 10.6: Alternating Series, Conditional Convergence

Fill in the blank for the various tests.

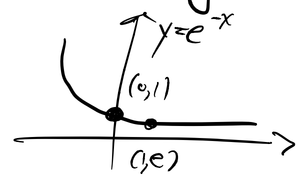
Office Hours 10:30 – noon Monday / Wednesday

E.g.: $\sum_{n=1}^{\infty} e^{-n}$ converge?

Integral Test: Need to find $f(x)$ such that

$$f(n) = a_n = e^{-n}, \quad f(x) = e^{-x} = \frac{1}{e^x}$$

- continuous on $[1, \infty)$
- decreasing: either observe $\frac{1}{e^x}$ is decreasing because e^x is increasing



or

$$f'(x) = -e^{-x} < 0 \text{ for all } x$$

- positive: because exponentials are always positive.

$\int_1^{\infty} f(x)dx$ converges if and only if $\sum_{n=1}^{\infty} a_n$ converges.

$u = -x$
$du = -dx$
$-du = dx$

$$\int_1^t e^{-x}dx = \int_{u(1)}^{u(t)} e^u(-du) = -\int_{-1}^{-t} e^u du = -e^u\Big|_{-1}^{-t}$$

$$= -(e^{-t} - e^{-1}) = e^{-1} - e^{-t}$$

$$\int_1^{\infty} e^{-x}dx = \lim_{t\to\infty} \int_1^t e^{-x}dx = \lim_{t\to\infty}\left(\frac{1}{e} - \frac{1}{e^t}\right) = \frac{1}{e} < \infty$$

$\int_1^\infty e^{-x}\,dx$ converges, so $\sum_{n=1}^{\infty} e^{-n}$ converges by the integral test.

E.g: $\sum_{n=1}^{\infty} \frac{1}{n}$ show diverges.

Hypothesis $f(x) = \frac{1}{x}$, continuous on $[1,\infty)$, decreasing, positive

Computation $\int_1^\infty \frac{dx}{x} = \lim_{t\to\infty} \int_1^t \frac{dx}{x} = \lim_{t\to\infty} \ln(x)\Big|_1^t = \lim_{t\to\infty} \ln(t) - \ln(1)$

$= \lim_{t\to\infty} \ln(t) = \infty$

Conclusion $\int_1^\infty \frac{dx}{x}$ diverges, so $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges by the Integral Test.

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$$ ← Know this

$$S_n = \frac{a(1-r^{n+1})}{1-r}$$ (Closed form for the partial sums)

E.g:

$$\sum_{n=1}^{\infty} \frac{n!}{10^n}$$ ← "Lots of multiplication"

Try ratio.

$$\lim_{n\to\infty} \underbrace{\frac{(n+1)!}{10^{n+1}}}_{a_{n+1}} \underbrace{\frac{10^n}{n!}}_{1/a_n} = \lim_{n\to\infty} \frac{n+1}{10} = \infty$$

Diverges by the Ratio Test.

E.g.: $\displaystyle\sum_{n=1}^{\infty} \frac{n\ln(n)}{(-2)^n}$ Try the Root Test

$$\lim_{n\to\infty} \sqrt[n]{\left|\frac{n\ln(n)}{(-2)^n}\right|} = \lim_{n\to\infty} \frac{\sqrt[n]{n}\,\sqrt[n]{\ln(n)}}{\sqrt[n]{2^n}}$$

$$= \lim_{n\to\infty} \frac{\sqrt[n]{n}\,\sqrt[n]{\ln(n)}}{2}$$

Know $\lim_{n\to\infty} \sqrt[n]{n} = 1$, need to know about

$$\lim_{n\to\infty} \sqrt[n]{\ln(n)} = \lim_{n\to\infty} \ln(n)^{1/n}$$

We can evaluate this using L'Hopital:

Rewrite $\ln(n) = e^{\ln(\ln(n))}$, so $\ln(n)^{1/n} = e^{\frac{1}{n}\ln(\ln(n))}$

and

$$\lim_{n\to\infty} \frac{\ln(\ln(n))}{n} \overset{\text{L'Hopital}}{=} \lim_{n\to\infty} \frac{\frac{1}{\ln(n)}\cdot\frac{1}{n}}{1} = \lim_{n\to\infty} \frac{1}{n\ln(n)} = 0$$

$$\Rightarrow \lim_{n\to\infty} \sqrt[n]{\ln(n)} = \lim_{n\to\infty} e^{\frac{1}{n}\ln(\ln(n))} = e^{\lim_{n\to\infty} \frac{\ln(\ln(n))}{n}} = e^0 = 1$$

$$\Rightarrow \lim_{n\to\infty} \sqrt[n]{\frac{n\ln(n)}{2^n}} = \lim_{n\to\infty} \frac{\sqrt[n]{n}\,\sqrt[n]{\ln(n)}}{2} = \frac{(1)(1)}{2} = \frac{1}{2} < 1,$$ so converges absolutely (Root Test).

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \quad \text{converges if } 1<p, \text{ diverges if } p\le 1.$$

$f(x) = 1/x^p$, continuous, decreasing, positive on $[1,\infty)$

$$\int_1^\infty \frac{dx}{x^p} = \lim_{t\to\infty} \int_1^t \frac{dx}{x^p}$$

<u>$p=1$</u>

$$\int_1^t \frac{dx}{x} = \ln(t) - \ln(1) = \ln(t) - 0 = \ln(t) \to \infty$$

as $t\to\infty$.

$p \neq 1$

$$\int_1^t x^{-p}\,dx = \left.\frac{x^{1-p}}{1-p}\right|_1^t = \frac{t^{1-p}}{1-p} - \frac{1}{1-p}$$

$$\lim_{t\to\infty} t^{1-p} = \begin{cases} \infty & \text{if } 1-p>0 \Leftrightarrow 1>p \\ 0 & \text{if } 1-p<0 \Leftrightarrow 1<p \end{cases}$$

$$\Rightarrow \int_1^\infty \frac{dx}{x^p} = \lim_{t\to\infty} \int_1^t \frac{dx}{x^p} = \lim_{t\to\infty}\left(\frac{t^{1-p}}{1-p} - \frac{1}{1-p}\right)$$

$$= \begin{cases} \infty & \text{if } 1\ge p \\ \frac{-1}{1-p} = \frac{1}{p-1} & \text{if } 1<p \end{cases}$$

By the integral test,

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \quad \text{converges if } 1<p \text{ and diverges if } p \le 1$$

The p-series are important for comparison tests.

E.g.: Does $\sum_{n=1}^{\infty} \sin(1/n)$ converge?

Recall: $\lim_{x \to 0} \frac{\sin(x)}{x} = 1.$

$$\lim_{n \to \infty} \frac{\sin(1/n)}{\underline{1/n}} = \lim_{h \to 0} \frac{\sin(h)}{h} = 1, \quad h = 1/n$$

Diverges by Limit Comparison with the Harmonic Series $\sum_{n=1}^{\infty} 1/n$ (p-series, $p=1$).