10.5: Ratio & Root Tests

3/6/18

Thm (The Ratio Test)

Let $\sum a_n$ be any series and suppose that

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \rho.$$

1) If $\rho < 1$, then the series converges absolutely.
2) If $\rho > 1$ or infinite, the series diverges.
3) If $\rho = 1$, then the test is inconclusive.

E.g: a) $\sum_{n=0}^{\infty} \frac{2^n+5}{3^n}$ b) $\sum_{n=1}^{\infty} \frac{(2n)!}{n!\,n!}$ c) $\sum_{n=1}^{\infty} \frac{4^n n!\,n!}{(2n)!}$

a) $$\lim_{n\to\infty} \frac{2^{n+1}+5}{3^{n+1}} \frac{3^n}{2^n+5} = \lim_{n\to\infty} \frac{1}{3} \frac{2^{n+1}+5}{2^n+5}$$

$$= \lim_{n\to\infty} \frac{1}{3} \frac{2^{n+1}(1+5/2^{n+1})}{2^n(1+5/2^n)}$$

$$= \lim_{n\to\infty} \frac{2}{3}\left(\frac{1+5/2^{n+1}}{1+5/2^n}\right)$$

$$= \frac{2}{3}\left(\frac{1+0}{1+0}\right) = 2/3 < 1$$

So $\sum_{n=0}^{\infty} \frac{2^n+5}{3^n}$ converges absolutely.

b) $\sum_{n=1}^{\infty} \frac{(2n)!}{n!\,n!}$

Observation $n! = n(n-1)(n-2)\cdots(2)(1)$

$$\frac{(n+1)!}{n!} = \frac{(n+1)\cancel{(n)}\cancel{(n-1)}\cancel{(n-2)}\cdots\cancel{(2)}\cancel{(1)}}{\cancel{n}\cancel{(n-1)}\cancel{(n-2)}\cdots\cancel{(2)}\cancel{(1)}}$$

$$= n+1.$$

$2(n+1) = 2n+2$

$$\lim_{n\to\infty} \frac{(2n+2)!}{(n+1)!\,(n+1)!} \frac{n!\,n!}{2n!} = \lim_{n\to\infty} \frac{(2n+2)(2n+1)\cancel{2n!}}{(n+1)(n+1)\cancel{2n!}}$$

$$= \lim_{n\to\infty} \frac{4n^2+6n+2}{n^2+2n+1}$$

$$= \frac{4}{1} = 4 > 1$$

By the Ratio Test $\sum_{n=1}^{\infty} \frac{(2n)!}{n!\,n!}$ diverges.

c) $\sum_{n=1}^{\infty} \frac{4^n n!\,n!}{(2n)!}$

$$\lim_{n\to\infty} \frac{4^{n+1}(n+1)!\,(n+1)!}{(2n+2)!} \frac{(2n)!}{4^n n!\,n!} = \lim_{n\to\infty} \frac{4(n+1)(n+1)}{(2n+2)(2n+1)}$$

$$= \lim_{n\to\infty} \frac{4n^2+8n+4}{4n^2+6n+2}$$

$$= \frac{4}{4} = 1$$

The Ratio Test is inconclusive!

$$\frac{a_{n+1}}{a_n} = \frac{4\cancel{(n+1)}(n+1)}{2\cancel{(n+1)}(2n+1)} = \frac{4(n+1)}{2(2n+1)} = \frac{2(n+1)}{2n+1}$$

$$\frac{a_{n+1}}{a_n} = \frac{2(n+1)}{2n+1} \geq \frac{2(1+1)}{2(1)+1} = \frac{4}{3} > 1$$

$$\Rightarrow a_{n+1} > a_n \geq a_1 = \frac{4(1!)(1!)}{(2(1))!} = \frac{4}{2} = 2$$

This tells us than $\lim_{n\to\infty} a_n \neq 0$. So the series

$$\sum_{n=1}^{\infty} \frac{4^n n! n!}{(2n)!}$$ diverges by the n^{th}-term test.

Thm (The Root Test)

Let $\sum a_n$ be any series and suppose that

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \rho$$

a) If $\rho < 1$, then the series converges absolutely.

b) If $1 < \rho$ or infinite, then the series diverges

c) If $\rho = 1$, the test is inconclusive.

E.g: Consider the sequence

$$a_n = \begin{cases} n/2^n & \text{if } n \text{ is odd,} \\ \frac{1}{2^n} & \text{if } n \text{ is even.} \end{cases}$$

Does $\sum_{n=0}^{\infty} a_n$ converge?

Consider the terms $\{a_{2n}\}_{n=0}^{\infty}$.

$$\lim_{n\to\infty} \sqrt[2n]{|a_{2n}|} = \lim_{n\to\infty} \sqrt[2n]{\frac{1}{2^{2n}}} = \lim_{n\to\infty} \left(\left(\frac{1}{2}\right)^{2n}\right)^{1/2n}$$

$$= 1/2$$

Consider the terms $\{a_{2n+1}\}_{n=0}^{\infty}$

$$\lim_{n\to\infty} \sqrt[2n+1]{|a_{2n+1}|} = \lim_{n\to\infty} \sqrt[2n+1]{\frac{2n+1}{2^{2n+1}}}$$

$$= \lim_{n\to\infty} \frac{\sqrt[2n+1]{2n+1}}{\sqrt[2n+1]{2^{2n+1}}}$$

$$= \lim_{n\to\infty} \frac{\sqrt[2n+1]{2n+1}}{2} = \frac{1}{2}$$

HW

If $a_{2n} \to L$ and $a_{2n+1} \to L$, then $a_n \to L$.

$\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1/2 < 1$, so

$\sum_{n=0}^{\infty} a_n$ converges by the Root Test.
(absolutely)

E.g.: a) $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ | $\lim_{n\to\infty} \sqrt[n]{n} = 1$, $\sqrt[n]{a^n} = a$, $a \geq 0$

$$\lim_{n\to\infty} \sqrt[n]{\frac{n^2}{2^n}} = \lim_{n\to\infty} \frac{\sqrt[n]{n^2}}{\sqrt[n]{2^n}} = \lim_{n\to\infty} \frac{(\sqrt[n]{n})^2}{2} = \frac{1}{2}$$

So $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ converges by the Root Test.

b) $\sum_{n=1}^{\infty} \frac{2^n}{n^3}$

$$\lim_{n\to\infty} \sqrt[n]{\frac{2^n}{n^3}} = \lim_{n\to\infty} \frac{2}{(\sqrt[n]{n})^3} = \frac{2}{1} = 2 > 1$$

Diverges by the Root Test.

c) $\sum_{n=1}^{\infty} \left(\frac{1}{1+n}\right)^n$

$$\lim_{n\to\infty} \sqrt[n]{\left(\frac{1}{1+n}\right)^n} = \lim_{n\to\infty} \frac{1}{1+n} = 0 < 1$$

So $\sum_{n=1}^{\infty} \left(\frac{1}{1+n}\right)^n$ converges by the Root Test.

10.6 Alternating Series and Conditional Convergence

Def$^{\underline{n}}$: A series in which the terms alternate signs is called an alternating series.

E.g: $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$

$1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \frac{1}{32} + \cdots$

Thm (Alternating Series Test)

The series

$$\sum_{n=1}^{\infty} (-1)^{n+1} u_n = u_1 - u_2 + u_3 - u_4 + \cdots$$

converges if the following three conditions are satisfied

1. $0 < u_n$ holds for all n,

2. $u_{n+1} \leq u_n$ holds for all $N \leq n$, for some integer N,

3. $u_n \to 0$

<u>E.g</u>: Alternating Harmonic Series

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$$

To apply the A.S.T $\quad u_n = 1/n$

✓ 1. $0 < 1/n$

✓ 2. $\frac{1}{n+1} \leq \frac{1}{n} \iff n \leq n+1$ holds for all n.

✓ 3. $\lim_{n \to \infty} \frac{1}{n} = 0$.

So
$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$
converges by the A.S.T.

Def$^{\underline{n}}$: We say a series $\sum a_n$ is conditionally convergent if $\sum a_n$ converges, but $\sum |a_n|$ diverges.

So the Alternating Harmonic Series is conditionally convergent because the Harmonic Series is Divergent.

Eg: $\sum_{n=1}^{\infty} (-1)^{n+1} \, 10n/n^2+16$ Does this converge?

1. $u_n = 10n/n^2+16 > 0$

3. $\lim_{n \to \infty} u_n = 0$ ✓

$$f(x) = 10x/x^2+16 \text{ decreasing } \Leftrightarrow$$

$$f'(x) < 0$$

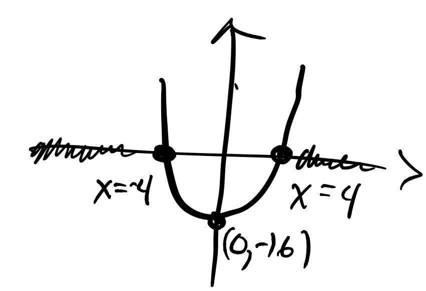
$$f'(x) = \frac{-10x^2+160}{(x^2+16)^2} < 0$$

$$\Leftrightarrow -10x^2+160 < 0$$

$$\Leftrightarrow -10(x^2-16) < 0$$

$$\Leftrightarrow x^2-16 > 0$$

$$\Leftrightarrow x < -4 \text{ or } x > 4$$



$u_{n+1} \le u_n$ when $5 \le n$.

So this series converges by the A.S.T.