10.4: Comparison Tests 3/1/18

Thm (The Comparison Test)

Let $\sum a_n$, $\sum c_n$, and $\sum d_n$ be series with non-negative terms. Suppose there exists some N such that

$$d_n \le a_n \le c_n \quad \text{when } N \le n$$

(a) If $\sum c_n$ converges, then $\sum a_n$ converges

(b) If $\sum d_n$ diverges, then $\sum a_n$ diverges.

E.g: $\displaystyle\sum_{n=1}^{\infty} \frac{5}{5n-1}$

$$\frac{5}{5n-1} = \frac{5}{5}\left(\frac{1}{n-1/5}\right) = \frac{1}{n-1/5} > \frac{1}{n}$$

$$\frac{1}{n} < \frac{1}{n-1/5} \iff n - 1/5 < n$$

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, so $\displaystyle\sum_{n=1}^{\infty} \frac{5}{5n-1}$ also diverges.

E.g: $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n!}$ $\quad [n! = n(n-1)(n-2)\cdots(2)(1),\ 0! = 1]$

Compare $\frac{1}{n!}$ to 2^n; Suspect this converges, So the inequality we want is

$$\frac{1}{n!} < \frac{1}{2^n}$$

since $\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n$ converges. Note that this inequality is equivalent to

$$2^n < n!$$

Observe

$$2^n = \underbrace{2 \cdot 2 \cdot 2 \cdots 2}_{n} \leq 2 \cdot 3 \cdot 4 \cdots n$$
$$= n!$$

at least for $3 < n$. So

$$\sum_{n=0}^{\infty} \frac{1}{n!}$$ converges by Comparison with $\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n$

E.g.: Consider the series

$$5 + 2/3 + 1/7 + 1 + \cdots + \frac{1}{2^n + \sqrt{n}} + \cdots$$
$$= 5 + 2/3 + 1/7 + 1 + \underline{\sum_{n=1}^{\infty} \frac{1}{2^n + \sqrt{n}}}$$

Observe that because $n \geq 1$

$$2^n \leq 2^n + \sqrt{n}$$
$$\Rightarrow \quad \frac{1}{2^n + \sqrt{n}} \leq \frac{1}{2^n} \leftarrow \text{terms of a convergent geometric series}$$

This says
$$\sum_{n=1}^{\infty} \frac{1}{2^n+\sqrt{n}} \quad \text{converges by comparison w/ } \sum_{n=1}^{\infty} \frac{1}{2^n}$$
So the original series
$$5+\tfrac{2}{3}+\tfrac{1}{7}+1+\sum_{n=1}^{\infty} \frac{1}{2^n+\sqrt{n}}$$
also converges.

<u>Thm</u> (Limit Comparison Test)

Suppose that $0<a_n$, $0<b_n$ hold for all $N \le n$ for some integer N.

1. If $\lim\limits_{n\to\infty} a_n/b_n = c > 0$, then both $\sum a_n$ and $\sum b_n$ converge or diverge.
2. If $\lim\limits_{n\to\infty} a_n/b_n = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges.
3. If $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.

Pf: (1) Choose $\varepsilon = c/2$ and N such that

$$\left|\frac{a_n}{b_n} - c\right| < \varepsilon = c/2$$

$$\Updownarrow$$

$$-\varepsilon < \frac{a_n}{b_n} - c < \varepsilon$$

$$\Updownarrow$$

$$c - \varepsilon = c - \frac{c}{2} = \frac{c}{2} < \frac{a_n}{b_n} < \varepsilon + c = \frac{c}{2} + c = \frac{3c}{2}$$

$\Rightarrow \frac{c}{2} b_n < a_n < \frac{3c}{2} b_n$.

If $\sum' b_n$ converges, then so does $\sum' \frac{3c}{2} b_n$ so by comparison $\sum' a_n$ converges.

If $\sum' b_n$ diverges, then $\sum' \frac{c}{2} b_n$ also diverges and $\sum' a_n$ diverges by comparison. $\blacksquare$

E.g.: a) $\sum'^{\infty}_{n=1} \frac{2n+1}{(n+1)^2}$, b) $\sum'^{\infty}_{n=1} \frac{1}{2^n - 1}$, c) $\sum'^{\infty}_{n=2} \frac{1 + n\ln(n)}{n^2+5}$

Compare a) to $\sum'^{\infty}_{n=1} \frac{1}{n}$

$$\lim_{n\to\infty} \frac{2n+1}{(n+1)^2} \Big/ \left(\frac{1}{n}\right) = \lim_{n\to\infty} \frac{n(2n+1)}{(n+1)^2}$$

$$= \lim_{n\to\infty} \frac{2n^2+n}{n^2+2n+1} = 2 > 0$$

L.C.T (1) says $\sum'^{\infty}_{n=1} \frac{2n+1}{(n+1)^2}$ diverges because $\sum'^{\infty}_{n=1} \frac{1}{n}$ diverges.

For $\sum_{n=1}^{\infty} \frac{1}{2^n-1}$, compare to $\sum_{n=1}^{\infty} \frac{1}{2^n}$

$$\lim_{n\to\infty} \left(\frac{1}{2^n-1}\right) \Big/ \left(\frac{1}{2^n}\right) = \lim_{n\to\infty} \frac{2^n}{2^n-1}$$

$$= \lim_{n\to\infty} \frac{2^n}{2^n}\left(\frac{1}{1-1/2^n}\right)$$

$$= \lim_{n\to\infty} \frac{1}{1-1/2^n} = \frac{1}{1-0} = 1 > 0$$

L.C.T. (1) says these both converge.

For $\sum_{n=2}^{\infty} \frac{1+n\ln(n)}{n^2+5}$

Compare to the Harmonic Series $\left(\sum_{n=1}^{\infty} \frac{1}{n}\right)$

$$\lim_{n\to\infty} \left(\frac{1+n\ln(n)}{n^2+5}\right) \Big/ \left(\frac{1}{n}\right) = \lim_{n\to\infty} \frac{n(1+n\ln(n))}{n^2+5}$$

$$= \lim_{n\to\infty} \frac{n^2(1/n+\ln(n))}{n^2(1+5/n^2)}$$

$$= \lim_{n\to\infty} \frac{\frac{1}{n}+\ln(n)}{1+5/n^2} = \infty$$

L.C.T (3) $\Rightarrow$

$\sum_{n=2}^{\infty} \frac{1+n\ln(n)}{n^2+5}$ diverges.

E.g.: Does

$$\sum_{n=1}^{\infty} \frac{\ln(n)}{n^{3/2}}$$

converge?

Know: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges when $1<p$

Choose $1 < p < 3/2$. Compare these two series using the L.C.T:

$$\lim_{n\to\infty} \frac{\ln(n)}{n^{3/2}} \Big/ \frac{1}{n^p} = \lim_{n\to\infty} \frac{n^p \ln(n)}{n^{3/2}}$$

$$= \lim_{n\to\infty} \frac{\ln(n)}{n^{3/2-p}}$$

Recall

$$\frac{n^p}{n^{3/2}} = n^{p-3/2} = \frac{1}{n^{-(p-3/2)}} = \frac{1}{n^{3/2-p}}$$

Note: $0 < 3/2 - p$

So $n^{3/2-p} \to \infty$, as does $\ln(n)$

Apply L'Hôpital

$$\lim_{n\to\infty} \frac{\frac{1}{n}}{(3/2-p)n^{3/2-p-1}} = \lim_{n\to\infty} \frac{1}{(3/2-p)n\,n^{3/2-p-1}}$$

$$= \lim_{n\to\infty} \frac{1}{(3/2-p)n^{3/2-p}} = 0.$$

L.C.T. (2) $\Rightarrow \displaystyle\sum_{n=1}^{\infty} \frac{\ln(n)}{n^{3/2}}$ converges.

10.5 Absolute Convergence; The Ratio and Root Tests.

Def$^{\underline{n}}$ A series $\sum a_n$ converges absolutely (is absolutely convergent) if

$$\sum |a_n| \text{ converges.}$$

E.g: (trivial) If $0 \le a_n$, and $\sum a_n$ converges then $\sum |a_n|$ because $a_n = |a_n|$.

E.g.: The series

$$5 - \frac{5}{4} + \frac{5}{16} - \frac{5}{64} + \cdots = \sum_{n=0}^{\infty} 5\left(\frac{-1}{4}\right)^n$$

is absolutely convergent because

$$\sum_{n=0}^{\infty} \left|5\left(\frac{-1}{4}\right)^n\right| = \sum_{n=0}^{\infty} 5\left(\frac{1}{4}\right)^n$$

is geometric with $r = \frac{1}{4} < 1$.

Thm (The Absolute Convergence Test)

If $\sum_{n=1}^{\infty} |a_n|$ converges, then so does $\sum_{n=1}^{\infty} a_n$.

Eg: $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^2} = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{6} + \cdots$

We know

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \sum_{n=1}^{\infty} \left| (-1)^{n+1} \frac{1}{n^2} \right|$$

converges, the A.C.T says

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^2} \text{ converges.}$$

<u>Caution</u>: It is <u>not</u> true that every series converges absolutely. The standard example is the Alternating Harmonic Series!

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \text{ converges}$$

but

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges}$$