Thm: Direct Comparison Test 2/27/18 1

Let f, g be continuous on $[a, \infty)$ with $0 \leq f(x) \leq g(x)$ for all $a \leq x$. Then

1. $\int_a^\infty f(x)\,dx$ converges if $\int_a^\infty g(x)\,dx$ converges,
2. $\int_a^\infty g(x)\,dx$ diverges if $\int_a^\infty f(x)\,dx$ diverges.

E.g.: Does $\int_1^\infty e^{-x^2}\,dx$ converge?

Observe $e^{-x^2} \leq e^{-x}$ for $1 \leq x$

$u = -x$

$du = dx$

$$\int_1^\infty e^{-x}\,dx = \lim_{t\to\infty} \int_1^{t} e^{-x}\,dx = \lim_{t\to\infty} -\int_{-1}^{-t} e^{u}\,du = \lim_{t\to\infty} -e^{u}\Big|_{-1}^{-t}$$

$$= \lim_{t\to\infty} -e^{-t} - (-e^{-1}) = \lim_{t\to\infty} \frac{1}{e} - \frac{1}{e^t} = \frac{1}{e}.$$

So by (1) above, $\int_1^\infty e^{-x^2}\,dx$ converges.

Thm: Limit Comparison Test

If the positive functions f and g are continuous on $[a,\infty)$ and if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L, \quad 0 < L < \infty$$

then

$$\int_a^\infty f(x)\,dx \quad \text{and} \quad \int_a^\infty g(x)\,dx$$

either both converge or both diverge.

E.g.: $\int_1^\infty \frac{dx}{1+x^2}$ converges by L.C.T w/ $g(x) = \frac{1}{x^2}$.

Compare to $\frac{1}{x^2}$: know that $\int_1^\infty \frac{dx}{x^2} < \infty$

$$\lim_{x\to\infty} \frac{\frac{1}{x^2+1}}{\frac{1}{x^2}} = \lim_{x\to\infty} \left(\frac{1}{x^2+1}\right)\left(\frac{x^2}{1}\right) = \lim_{x\to\infty} \frac{x^2}{x^2+1}$$

$$= \lim_{x\to\infty} \frac{x^2}{x^2}\left(\frac{1}{1+1/x^2}\right) = \lim_{x\to\infty} \frac{1}{1+\frac{1}{x^2}} = \frac{1}{1} = 1.$$

10.3: The Integral Test

Non-Decreasing Partial Sums

Suppose we have $0 \leq a_n$ for all n. The partial sums are all non-decreasing because

$$S_n = S_n + 0 \leq S_n + a_{n+1} = S_{n+1}$$

So we have the following corollary to the Monotone Sequence Theorem:

A series $\sum a_n$ of non-negative terms $(0 \leq a_n)$ converges if and only if its partial sums are bounded above.

Thm Let $\{a_n\}$ be a sequence of positive terms. Suppose that f is a continuous, positive, decreasing function of x for all $N \leq x$ for some positive integer N.

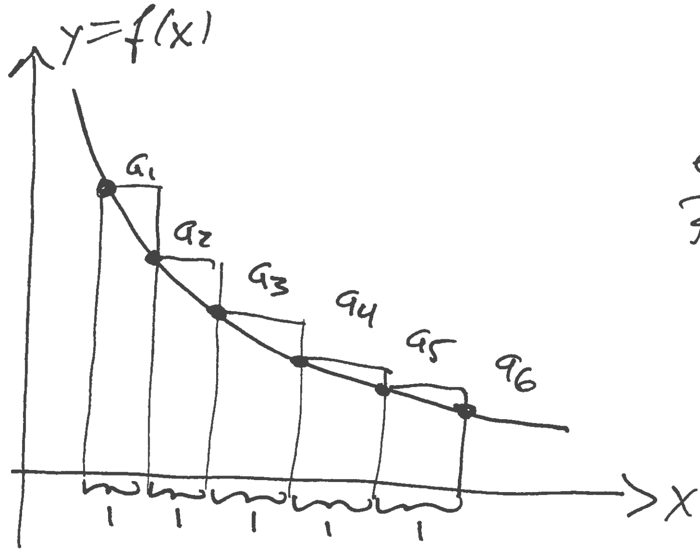
If $f(n) = a_n$ for each integer $N \leq n$, then

$$\sum_{n=N}^{\infty} a_n \quad \text{and} \quad \int_N^{\infty} f(x)\,dx$$

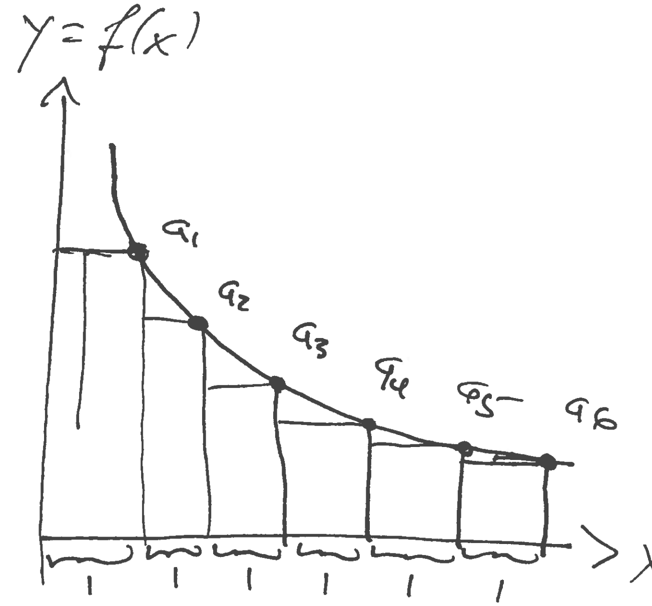
either both converge or both diverge.

Pf (sketch): Assume $N = 1$.

Left endpoint Riemann Sum



Right endpoint Riemann Sum



Partition on left

Partition on right is the same.

$$x_0 = 0 < x_1 = 1 < x_2 = 2 < \cdots$$

$\Delta x = 1.$

$f(x_i) = a_i, \; i \geq 1$

$$\sum_{n=1}^{\infty} f(x_n)\,\Delta x = \sum_{n=1}^{\infty} a_n \quad \text{over-estimate}$$

$$\sum_{n=1}^{\infty} f(x_n)\,\Delta x = \sum_{n=1}^{\infty} a_n. \quad \text{under-estimate}$$

$$\int_1^{\infty} f(x)\,dx \leq \sum_{n=1}^{\infty} a_n$$

$$a_1 + \int_1^{\infty} f(x)\,dx \geq \sum_{n=1}^{\infty} a_n$$

Eg: The Series

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

converges if and only if $p>1$. Back in §8.8, we showed that

$$\int_1^{\infty} \frac{dx}{x^p} = \begin{cases} \frac{1}{p-1} & \text{if } 1<p, \\ \infty & \text{if } p \leq 1 \end{cases}$$

$f(x) = \frac{1}{x^p}$: continuous, decreasing, satisfies $f(n) = \frac{1}{n^p}$ $n \geq 1$.

Apply Integral Test.

Eg: $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ converges by the Integral Test because

$$\int_1^{\infty} \frac{dx}{x^2+1} < \infty.$$

Eg: $\sum_{n=1}^{\infty} e^{-n^2}$ converges because $\int_1^{\infty} e^{-x^2}dx$ converges.

Eg: The series

$$\sum_{n=1}^{\infty} \frac{1}{2^{\ln(n)}}$$

diverges.

$$2^{\ln(n)} = \left(e^{\ln(2)}\right)^{\ln(n)} = e^{\ln(2)\ln(n)} = e^{\ln(n)\ln(2)}$$

$$= \left(e^{\ln(n)}\right)^{\ln(2)} = n^{\ln(2)}$$

Know $2 < e$ and $\ln(x)$ is increasing, so

$$\ln(2) < \ln(e) = 1$$

$$\frac{1}{2^{\ln(n)}} = \frac{1}{n^{\ln(2)}} \Rightarrow \sum_{n=1}^{\infty} \frac{1}{2^{\ln(n)}} = \infty.$$

Error Estimation

The number $R_n = a_{n+1} + a_{n+2} + \dots$ for some convergent series $\sum_{n=1}^{\infty} a_n$ can be thought of as

$$R_n = \sum_{n=1}^{\infty} a_n - S_n \quad (n^{\text{th}} \text{ remainder})$$

Is the error from estimating the sum of the series $\left(\sum_{n=1}^{\infty} a_n\right)$ by the n^{th} partial sum ($S_n = a_1 + a_2 + \dots + a_n$). We have the inequalities

$$\int_{n+1}^{\infty} f(x)\,dx \le R_n \le \int_{n}^{\infty} f(x)\,dx$$

coming from the bounds in the proof of the integral test

Eg.: Consider $\sum_{n=1}^{\infty} \frac{1}{n^2} = \pi^2/6$. Estimate the sum using 10 terms; check the error. The error is bounded by:

$$\int_{11}^{\infty} \frac{dx}{x^2} \le R_{10} \le \int_{10}^{\infty} \frac{dx}{x^2}$$

$$\int_n^t x^{-2}\,dx = (-1)x^{-1}\Big|_n^t = -t^{-1} - (-n)^{-1} = \frac{1}{n} - \frac{1}{t}.$$ ⑦

$$\int_n^\infty x^{-2}\,dx = \lim_{t\to\infty} \int_n^t x^{-2}\,dx = \lim_{t\to\infty}\left(\frac{1}{n} - \frac{1}{t}\right) = \frac{1}{n}.$$

$$\Rightarrow \int_{11}^\infty \frac{dx}{x^2} = \frac{1}{11}; \quad \int_{10}^\infty \frac{dx}{x^2} = \frac{1}{10}, \text{ So } \underline{\frac{1}{11} \le R_{10} \le \frac{1}{10}}.$$

$$S_{10} = \frac{1}{1^2} + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36} + \frac{1}{49} + \frac{1}{64} + \frac{1}{81} + \frac{1}{100}$$

$$\underline{\approx 1.54977}$$

Also remember that

$$\frac{1}{11} \le R_{10} = \sum_{n=1}^\infty \frac{1}{n^2} - S_{10} \le \frac{1}{10}$$

$$\frac{1}{11} + S_{10} \le \sum_{n=1}^\infty \frac{1}{n^2} \le \frac{1}{10} + S_{10}$$

$$\begin{array}{ccc} \| & & \| \\ 1.64068 & & 1.64977 \end{array}$$

Could use the midpoint, ≈ 1.6452, to estimate $\sum_{n=1}^\infty \frac{1}{n^2} = \frac{\pi^2}{6} \approx 1.64493$.