The n^{th} Term Test For Divergence

Thm: If $\sum_{n=1}^{\infty} a_n = L$, then $\lim_{n\to\infty} a_n = 0$.

an equivalent ~~phrasing~~ way to say this is

If $\lim_{n\to\infty} a_n \neq 0$ or does not exist, then $\sum_{n=1}^{\infty} a_n$ is divergent.

This is the contrapositive.

Lemma: If $\lim_{n\to\infty} a_n = L$, then for every $0<\varepsilon$, there exists some $0 \leq N$ such that

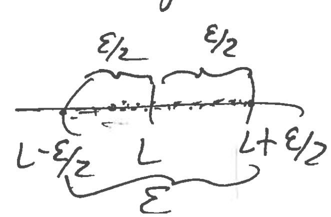
$$|a_n - a_m| < \varepsilon$$

holds for all $n, m \geq N$.

Pf: By def^n there exists some N such that given $0<\varepsilon$

$$|a_n - L| < \varepsilon/2$$

whenever $n \geq N$.



So when $m,n \geq N$

$$|a_n - a_m| = |a_n - a_m + L - L|$$
$$= |a_n - L + L - a_m|$$
$$\leq |a_n - L| + |L - a_m|$$
$$< \varepsilon/2 + \varepsilon/2 = \varepsilon. \quad \blacksquare$$

Pf (Thm): $\sum_{n=1}^{\infty} a_n = L = \lim_{n\to\infty} S_n$ (definition of convergence of Series)

By the lemma, we can choose some $N \geq 0$ such that whenever $N \leq n-1$

$$|a_n| = |\underbrace{a_1 + a_2 + a_3 + \dots + a_n}_{S_n} - (\underbrace{a_1 + a_2 + a_3 + \dots + a_{n-1}}_{S_{n-1}})|$$
$$= |S_n - S_{n-1}| < \varepsilon$$

So by definition, $a_n \to 0$. $\blacksquare$

<u>Warning</u>: It is <u>not true</u> in general that if
$a_n \to 0$, then $\sum_{n=1}^{\infty} a_n$ converges.

As an example, the <u>Harmonic</u> <u>Series</u>

$\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, even though $\frac{1}{n} \to 0$

Pf: Assume to the contrary that $\sum_{n=1}^{\infty} \frac{1}{n} = L$.

$$\begin{aligned}
L &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \cdots \\
&\geq 1 + \frac{1}{2} + \underbrace{\frac{1}{4} + \frac{1}{4}} + \underbrace{\frac{1}{6} + \frac{1}{6}} + \underbrace{\frac{1}{8} + \frac{1}{8}} + \cdots \\
&= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots \\
&= \frac{1}{2} + \underbrace{1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots}_{L} \\
&= \frac{1}{2} + L
\end{aligned}$$

So $L \geq \frac{1}{2} + L$, thus $0 \geq \frac{1}{2}$ is a contradiction. Therefore no such L exists! ∎

E.g.: 1. $\sum_{n=1}^{\infty} n^2$ diverges because $\lim_{n\to\infty} n^2 = \infty$.

2. $\sum_{n=1}^{\infty} \frac{n}{n+1}$ diverges because $\lim_{n\to\infty} \frac{n}{n+1} = 1$.

3. $\sum_{n=1}^{\infty} (-1)^{n+1}$ diverges because $\lim_{n\to\infty} (-1)^{n+1}$ does not exist.

4. $\sum_{n=1}^{\infty} \frac{-n}{2n+5}$ diverges because $\lim_{n\to\infty} \frac{-n}{2n+5} = \frac{-1}{2}$.

Arithmetic With Series

Thm: If $\sum_{n=1}^{\infty} a_n = A$, $\sum_{n=1}^{\infty} b_n = B$, then

1. $\sum_{n=1}^{\infty} (a_n + b_n) = A + B$
2. $\sum_{n=1}^{\infty} (a_n - b_n) = A - B$
3. $\sum_{n=1}^{\infty} k a_n = kA$

k a constant.

<u>Corollary</u>: ① Every non-zero constant multiple of a divergent series diverges.

② If $\sum a_n$ converges and $\sum b_n$ diverges, then $\sum (a_n+b_n)$ diverges.

Pf: ① Assume $\sum b_n$ diverges, but $\sum kb_n$ converges, some $k \neq 0$. By Part ③ of the theorem,

$$\sum \frac{1}{k}(kb_n) = \sum b_n$$

converges, a contradiction.

② $\sum a_n$ converges, $\sum b_n$ diverges; assume to the contrary that $\sum (a_n+b_n)$ converges. Then $\sum -a_n$ converges by pt ③ and by part ①, $\sum [a_n + (a_n+b_n)] = \sum b_n$ converges, a contradiction. ∎

<u>Warning</u>: If $\sum a_n$ and $\sum b_n$ are both divergent, ② is <u>not</u> necessarily true.

E.g.: $a_n = \frac{1}{n}$, $b_n = \frac{-1}{n}$, $\sum \frac{1}{n}$ diverges, by ① $\sum \frac{-1}{n}$ diverges, but

$$\sum (a_n+b_n) = \sum \frac{1}{n} - \frac{1}{n} = \sum 0 = 0.$$

Eg: Find the sum of

a) $\sum_{n=1}^{\infty} \frac{3^{n-1}-1}{6^{n-1}}$

$$\frac{3^{n-1}-1}{6^{n-1}} = \frac{3^{n-1}}{6^{n-1}} - \frac{1}{6^{n-1}}$$

$$= \left(\frac{3}{6}\right)^{n-1} - \left(\frac{1}{6}\right)^{n-1}$$

$$= \left(\frac{1}{2}\right)^{n-1} - \left(\frac{1}{6}\right)^{n-1}$$

$$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots = 2 = \frac{10}{5}$$

b) $\sum_{n=0}^{\infty} \frac{4}{2^n} = 4 \cdot 2 = 8.$

because $\sum_{n=0}^{\infty} \frac{1}{2^n} = 2$

$a=1, r=1/6$

$$\sum_{n=1}^{\infty} \left(\frac{1}{6}\right)^{n-1} = \frac{1}{1-1/6} = \frac{1}{5/6} = \frac{6}{5}$$

$$\sum_{n=1}^{\infty} \frac{3^{n-1}-1}{6^{n-1}} = \sum_{n=1}^{\infty} \left(\left(\frac{1}{2}\right)^{n-1} - \left(\frac{1}{6}\right)^{n-1}\right)$$

$$= \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1} - \sum_{n=1}^{\infty} \left(\frac{1}{6}\right)^{n-1}$$

$$= \frac{10}{5} - \frac{6}{4} = \frac{4}{5}.$$

$$a + ar + ar^2 + ar^3 + \cdots$$

$$S_n = a + ar + ar^2 + \dots + ar^n$$

$$rS_n = \quad ar + ar^2 + \dots + ar^n + ar^{n+1}$$

$$(1-r)S_n = S_n - rS_n = a - ar^{n+1} = a(1-r^{n+1})$$

$$S_n = \frac{a(1-r^{n+1})}{1-r} \longrightarrow \frac{a}{1-r} \text{ as } n \to \infty. \text{ when } |r|<1$$

Reindexing a series

So long as you don't change the order of the terms, you can always change the starting point of the series. To move it h units replace n in the sum by $n-h$

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + a_4 + \cdots$$

$$= \sum_{n=1+h}^{\infty} a_{n-h}$$

Eg: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^{n-1}} = \sum_{n=0}^{\infty} \frac{1}{2^n} = \sum_{n=5}^{\infty} \frac{1}{2^{n-5}}$

Adding or Deleting Terms

Adding or deleting any finite number of terms does not have any effect on the convergence or the divergence.

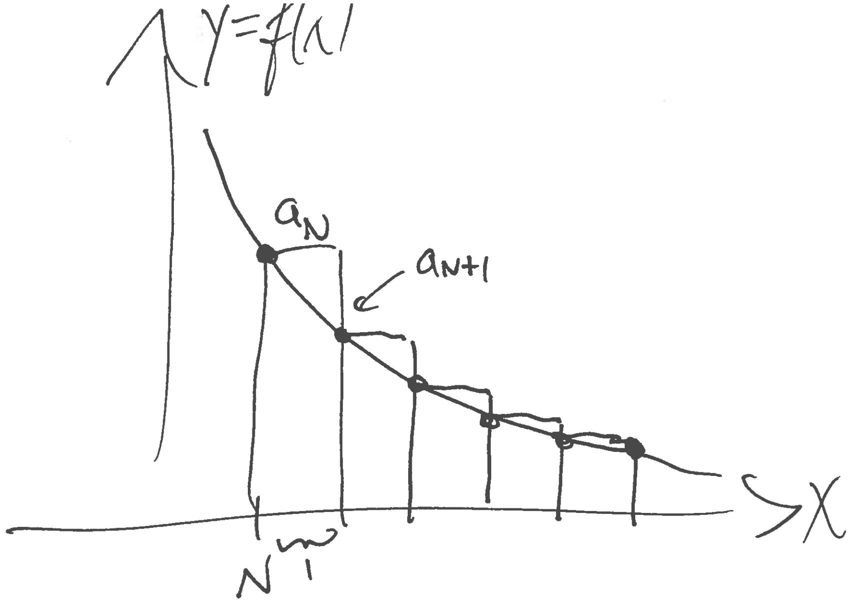
$$\sum_{n=1}^{\infty} a_n = \underline{a_1 + a_2 + a_3 + \cdots + a_{N-1}} + \sum_{n=N}^{\infty} a_n$$

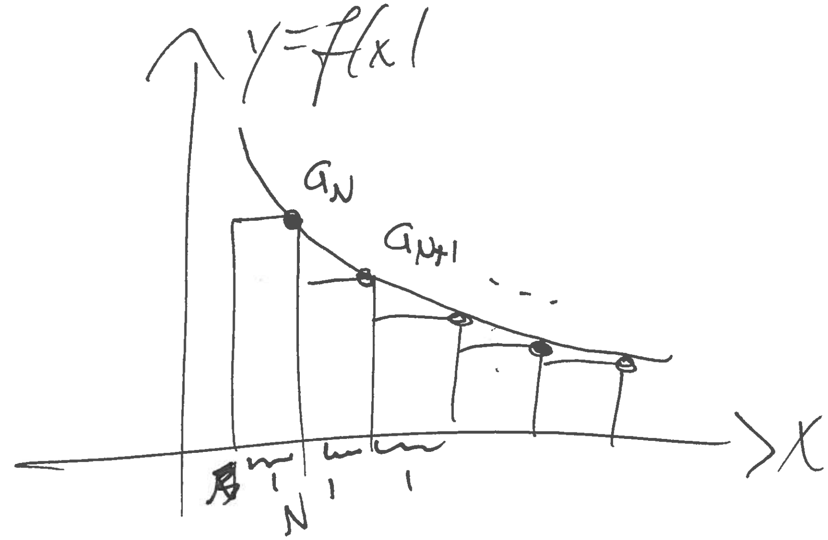
$\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=N}^{\infty} a_n$ converges.

Rmk: The sum of $\sum_{n=1}^{\infty} a_n$ is not the sum of $\sum_{n=N}^{\infty} a_n$.

10.3 The Integral Test

Thm: Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of positive terms ($0 \lneq a_n$) and assume that f is a positive, continuous, decreasing function of x for all $N \leq x$. ($N \geq 0$ an integer). Suppose that for all integers n, $f(n) = a_n$. Then $\sum_{n=N}^{\infty} a_n$ and the improper integral $\int_N^{\infty} f(x)dx$ either both converge or both diverge.


$y=f(x)$
a_N
a_{N+1}
x
N
1


$y=f(x)$
a_N
a_{N+1}
...
x
N
1
1
1