<u>Thm</u> (Squeeze): Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $a_n \le b_n \le c_n$, for all $N \le n$ and if $a_n \to L$, $c_n \to L$, then $b_n \to L$.

<u>Cor</u>: If $|b_n| \le c_n$ and $c_n \to 0$, then $b_n \to 0$.

Pf: Observe that $|b_n| \le c_n$ means $-c_n \le b_n \le c_n$, apply Squeeze Theorem with $a_n = -c_n$. ∎

<u>E.g.</u>: $\frac{\cos(n)}{n} \to 0$ because $\left|\frac{\cos(n)}{n}\right| = \frac{|\cos(n)|}{n} \le \frac{1}{n}$ and $\frac{1}{n} \to 0$.

E.g.: $\frac{(-1)^n}{n} \to 0$ because $\frac{|(-1)^n|}{n} \le \frac{1}{n} \to 0$

<u>Thm</u>: Let $\{a_n\}$ be a sequence of real numbers. If $a_n \to L$ and f is continuous at L, then
$$\lim_{n\to\infty} f(a_n) = f\left(\lim_{n\to\infty} a_n\right) = f(L).$$

E.g.: Show that

$$\lim_{n\to\infty} \frac{\ln(n)}{n} = 0$$

Since $\ln(x)$ and x are both continuous functions,

$$f(x) = \frac{\ln(x)}{x}$$

is continuous away from zero.

$$\lim_{n\to\infty} \cancel{f(x)}\, \frac{\ln(n)}{n} = \lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{\ln(x)}{x}$$

$$\overset{\text{L'Hôpital}}{=} \lim_{x\to\infty} \frac{\left(\frac{1}{x}\right)}{1} = \lim_{x\to\infty} \frac{1}{x} = 0.$$

Thm: 1. $\lim_{n\to\infty} \frac{\ln(n)}{n} = 0$ 2. $\lim_{n\to\infty} \sqrt[n]{n} = 1$

3. $\lim_{n\to\infty} x^{1/n} = 1, \ x > 0$ 4. $\lim_{n\to\infty} x^n = 0, \ |x| < 1$

5. $\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x$ 6. $\lim_{n\to\infty} \frac{x^n}{n!} = 0.$

Def^n: A sequence defined <u>recursively</u> gives some initial terms, then describes how to compute later terms from these.

E.g.: $a_1 = 1$, $a_n = a_{n-1} + 1$.

$a_1 = 1$, $a_2 = 1+1 = 2$, $a_3 = 2+1 = 3$, ...

E.g.: (Fibonacci) $a_1 = 1$, $a_2 = 1$, $a_{n+1} = a_n + a_{n-1}$, $n \geq 2$.

$a_1 = 1$, $a_2 = 1$, $a_3 = 2$, $a_4 = 3$, $a_5 = 5$, ...

Def^n: A sequence $\{a_n\}$ is said to be

1. <u>bounded above</u> if $a_n \leq M$ for all n and some M,
2. <u>bounded below</u> if $M \leq a_n$ for all n and some M,
3. <u>bounded</u> if bounded above & below.

Eg: ① 1, 2, 3, 4, ... is bounded below, but not above
② $a_n = \frac{n}{n+1}$ is bounded above by 1, bounded below by 0.

If $a_n \le M$ for all n, we say M is an upper bound for the sequence. The least upper bound (lub) or supremum (sup) is the smallest upper bound.

If $m \le a_n$ for all n, we say m is a lower bound for the sequence. The greatest lower bound (glb) or infimum (inf) is the largest lower bound.

E.g. The infimum of $1, 2, 3, \ldots$ is 1.

The supremum of $\frac{n}{n+1}$ is 1.

Def^n: A sequence $\{a_n\}$ is called

1. non-decreasing if $a_n \le a_{n+1}$ for all n,
2. non-increasing if $a_{n+1} \le a_n$ for all n,
3. monotone if it is either one of these.

Thm Every bounded montone sequence converges.

10.2 : Infinite Series

Definiton: Given a sequence of numbers, $\{a_n\}_{n=1}^{\infty}$, an expression of the form

$$\sum_{n=1}^{\infty} a_n := a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

is an infinite series.

The n^{th} partial sum is

$$S_n = a_1 + a_2 + a_3 + \cdots + a_n .$$

which form a sequence of partial sums, $\{S_n\}_{n=1}^{\infty}$. We say that the infinite series converges to L if

$$\lim_{n\to\infty} S_n = L$$

and we write

$$\sum_{n=1}^{\infty} a_n = L.$$

Otherwise, the series diverges.

Geometric Series

A geometric series has the form

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots + ar^{n-1} + \cdots$$

$a \neq 0$, $1 \neq r$ real numbers.

There's a clever way to determine the formula for the partial sums:

$$S_n = a + ar + ar^2 + \cdots + ar^{n-1}$$

$$rS_n = \quad ar + ar^2 + \cdots + ar^{n-1} + ar^n.$$

$$(1-r)S_n = S_n - rS_n = a - ar^n = a(1-r^n)$$

$$\Rightarrow S_n = \frac{a(1-r^n)}{1-r}.$$

We note that if $|r|<1$, then $\lim_{n\to\infty} r^n = 0$, and $\lim_{n\to\infty} r^n = \infty$ if $1<|r|$

So

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a(1-r^n)}{1-r} = \begin{cases} \infty & \text{if } |r|>1 \\ \frac{a}{1-r} & \text{if } |r|<1. \end{cases}$$

Eg.: $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$

$$\left(\tfrac{1}{2}\right)^0 \quad \left(\tfrac{1}{2}\right)^1 \quad \left(\tfrac{1}{2}\right)^2 \quad \left(\tfrac{1}{2}\right)^3 \quad \left(\tfrac{1}{2}\right)^4$$

$$\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n = \sum_{n=1}^{\infty} 1\left(\frac{1}{2}\right)^{n-1} = \frac{1}{1-\frac{1}{2}} = \frac{1}{\frac{1}{2}} = 2.$$

Eg: $0.\overline{9} = 0.99999\cdots = 1.$

$$0.\overline{9} = \frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \frac{9}{10000} + \frac{9}{100000} + \cdots$$
$$= \frac{9}{10} + \frac{9}{10^2} + \frac{9}{10^3} + \frac{9}{10^4} + \frac{9}{10^5} + \cdots$$

$$= \frac{9}{10}(1) + \frac{9}{10}\left(\frac{1}{10}\right) + \frac{9}{10}\left(\frac{1}{10^2}\right) + \frac{9}{10}\left(\frac{1}{10^3}\right) + \frac{9}{10}\left(\frac{1}{10^4}\right) + \cdots$$

$$= \sum_{n=1}^{\infty} \frac{9}{10}\left(\frac{1}{10}\right)^{n-1} \quad \text{geometric, } a = 9/10,\ r = 1/10$$

$$= \frac{9/10}{1 - 1/10} = \frac{\frac{9}{10}}{\frac{9}{10}} = 1.$$

E.g.: Find the sum of the "telescoping" series

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}.$$

$$\frac{1}{n(n+1)} = \frac{(n+1) - n}{n(n+1)} = \frac{n+1}{n(n+1)} - \frac{n}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

$$S_n = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n}\right) + \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

0 0 0 0 0

$$= 1 - \frac{1}{n+1} \to 1 \quad \text{as } n \to \infty.$$

$$\text{So } \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \lim_{n\to\infty} S_n = 1.$$