10.1: Sequences

Def$^{\underline{n}}$: A **sequence** is a list of numbers

$$a_1, a_2, a_3, a_4, \ldots, a_n, \ldots$$

The ~~ } ~~ subscript is called the index

You can think of this as a function a with domain the integers ≥ 1, and target the real numbers $a(n) = a_n$.

E.g.: $\{a_n = 10+n\}_{n=1}^{\infty}$ is the sequence

$$a_1 = 11,\ a_2 = 12,\ a_3 = 13,\ a_4 = 14, \ldots$$

$\{\sqrt{n}\}_{n=1}^{\infty}$ is the sequence

$$1, \sqrt{2}, \sqrt{3}, 2, \sqrt{5}, \sqrt{6}, \sqrt{7}, \sqrt{8}, 3, \ldots$$

The natural question given a sequence is "does this have a limit?"

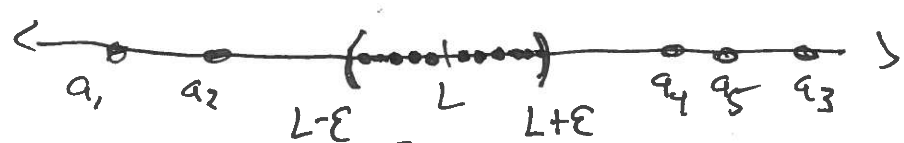
Def$^{\underline{n}}$: The sequence, $\{a_n\}$, converges to the number L if for every $0<\varepsilon$ there exists an integer N such that for all $N \leq n$

$$|a_n - L| < \varepsilon$$

If no such L exists, then we say the sequence diverges.

If $\{a_n\}_{n=1}^{\infty}$ converges to L, then we write $\lim_{n\to\infty} a_n = L$, or simply $a_n \to L$, and we call L the <u>limit</u> <u>of</u> <u>the</u> <u>sequence</u>.

The cartoon sketch of convergence is



all of a_n's for $N \le n$ live in this region

<u>Rmk:</u>

$|a_n - L| < \varepsilon$ by definition means

$$-\varepsilon < a_n - L < \varepsilon$$

which is equivalent to saying

$$L-\varepsilon < a_n < L+\varepsilon.$$

E.g.: $\left\{a_n = \frac{1}{10^n}\right\}_{n=1}^{\infty}$ This sequence converges to zero.

Pf: For given $\varepsilon > 0$, we need to choose some N such that whenever $N \le n$

$$\left|\frac{1}{10^n}\right| = \left|\frac{1}{10^n} - 0\right| < \varepsilon \iff -\varepsilon < \frac{1}{10^n} < \varepsilon \iff \frac{1}{10^n} < \varepsilon$$

If we choose $-\log_{10}(\varepsilon) < N$, then we have our inequality.

$$\frac{1}{10^N} < \varepsilon \iff 1 < \varepsilon 10^N \iff \frac{1}{\varepsilon} < 10^N \iff \underbrace{\log_{10}\left(\frac{1}{\varepsilon}\right)}_{= -\log_{10}(\varepsilon)} < \log_{10}(10^N) = N$$

E.g.: Show $\lim_{n \to \infty} \frac{1}{n} = 0$.

We want to choose for any $\varepsilon > 0$, some N such that

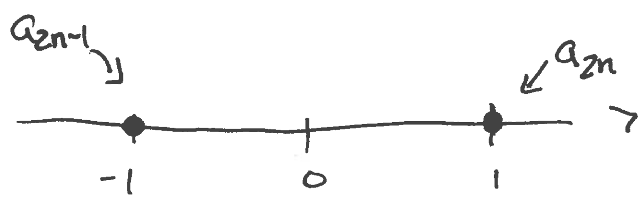
$$\left| \frac{1}{N} - 0 \right| < \varepsilon$$

$$\left| \frac{1}{N} - 0 \right| = \left| \frac{1}{N} \right| = \frac{1}{N} < \varepsilon \iff 1 < \varepsilon N \iff \boxed{\frac{1}{\varepsilon} < N}$$

For any $\frac{1}{\varepsilon} < n$ we have

$$\frac{1}{n} = \left| \frac{1}{n} \right| < \varepsilon.$$

E.g.: Show that sequence $\{(-1)^n\}_{n=1}^{\infty}$ diverges.



Suppose that there exists some L such that for $\varepsilon = \frac{1}{2}$, there exists some L such that for all $N \le n$

$$\left| (-1)^n - L \right| < \varepsilon = \tfrac{1}{2} \iff -\tfrac{1}{2} + (-1)^n < L < \tfrac{1}{2} + (-1)^n$$

This says that (because this hold for all N by assumption)

$$\frac{1}{2} = -\frac{1}{2} + 1 < L < \frac{1}{2} + 1 = 3/2 \qquad (n \text{ even}, N \leq n)$$

$$-\frac{3}{2} = -\frac{1}{2} - 1 < L < \frac{1}{2} - 1 = -\frac{1}{2} \qquad (n \text{ odd}, N \leq n)$$

This implies L is positive and L is negative; this is absurd, So such L exists. ∎

Def^{n}: The sequence $\{a_n\}_{n=1}^{\infty}$ <u>diverges</u> <u>to</u> <u>infinity</u> if for every number M, there exists an integer N such that for all $N \leq n$,

$$M \lneq a_n$$

In this case we write

$$\lim_{n \to \infty} a_n = \infty.$$

We say that $\{a_n\}_{n=1}^{\infty}$ <u>diverges</u> <u>to</u> <u>negative</u> <u>infinity</u>, if for every positive integer M, there exists an N such that

$$a_n \lneq -M \quad \text{for all } N \leq n$$

and we write

$$\lim_{n \to \infty} a_n = -\infty.$$

E.g.: $\{\sqrt{n}\}_{n=1}^{\infty}$ diverges to infinity

Pf: Given some $0 < M$ (integer). We want to find some N such that

$$M < \sqrt{N} \iff M^2 < N \quad (\text{because } f(x) = x^2 \text{ is increasing on } [0,\infty))$$

whenever $N \leq n$ we have

$$M < \sqrt{N} < \sqrt{n} \quad (\text{because } f(x) = \sqrt{x} \text{ is increasing on } [0,\infty).)$$

provided $M^2 < N$.

Thm: Let $\{a_n\}$, $\{b_n\}$ be sequences of real numbers, and let

$$\lim_{n\to\infty} a_n = A, \quad \lim_{n\to\infty} b_n = B,$$

then

① $\lim_{n\to\infty} (a_n + b_n) = A + B$

② $\lim_{n\to\infty} (a_n - b_n) = A - B$

③ $\lim_{n\to\infty} k a_n = kA$, k constant

④ $\lim_{n\to\infty} (a_n b_n) = AB$

⑤ $\lim_{n\to\infty} \left(\frac{a_n}{b_n}\right) = \frac{A}{B}$.

E.g: $\lim_{n\to\infty} \left(\frac{-1}{n}\right) = \lim_{n\to\infty} (-1) \lim_{n\to\infty} \frac{1}{n} = (-1)(0) = 0.$

$$\lim_{n\to\infty}\left(\frac{n-1}{n}\right) = \lim_{n\to\infty}\left(\frac{n}{n}-\frac{1}{n}\right) = \lim_{n\to\infty}\left(1-\frac{1}{n}\right) = 1-0=1.$$

$$\lim_{n\to\infty}\left(\frac{4-7n^6}{n^6+3}\right) = \lim_{n\to\infty}\left(\frac{n^6}{n^6}\frac{(4/n^6-7)}{(1+3/n^6)}\right) = \lim_{n\to\infty}\left(\frac{4/n^6-7}{1+3/n^6}\right) = \frac{-7}{1} = -7.$$

Rmk: Be warned that none of these things are true if either $\{a_n\}$ or $\{b_n\}$ (or both) diverge.

E.g.: $a_n = n$, $b_n = -n$. $\lim a_n = \infty$, $\lim b_n = -\infty$

$$\lim(a_n+b_n) = \lim(n+-n) = \lim(0) = 0$$

This might lead you to believe that

$$\lim a_n + \lim b_n = \infty + -\infty = 0.$$

This is very wrong.

Consider $a_n = n^2$, $b_n = -n$. $\lim a_n = \infty$, $\lim b_n = -\infty$

But $\lim(a_n+b_n) = \lim(n^2-n) = \infty$, not 0.

$$f(x) = x^2 - x = x(x-1)$$