# MATH 142: SUPPLEMENTARY EXAMPLE

BLAKE FARMAN
UNIVERSITY OF SOUTH CAROLINA

## 1. A Class of Improper Integrals of Type II

This note is concerned primarily with a geometric observation about the convergence of Improper Integrals of Type II of the form

$$\int_0^1 \frac{\mathrm{d}x}{x^p}$$

stated in terms of certain Improper Integrals of Type I. We observe that when $p \leq 0$, the function $1/x^p$ is defined at zero (e.g., say $p = -2$, then $1/x^{-2} = x^2$), so when discussing Improper Integrals of Type II, we will assume that $0 < p$.

For the integral of interest, one can compute the values of $p$ for which we have convergence directly. When $p = 1$ we have

$$\int_0^1 \frac{dx}{x} = \lim_{t\to 0^+} \int_t^1 \frac{dx}{x} = \lim_{t\to 0^+} \ln(x)\Big|_t^1 = \lim_{t\to 0^+} (0 - \ln(t)) = \lim_{t\to 0^+} -\ln(t) = \infty$$

and for $p \neq 1$ we have

$$\int_0^1 \frac{dx}{x^p} = \lim_{t\to 0^+} \int_t^1 x^{-p}\,\mathrm{d}x = \lim_{t\to 0^+} \frac{x^{1-p}}{1-p}\Big|_t^1 = \lim_{t\to 0^+} \left(\frac{1}{1-p}\right)\left(1 - t^{1-p}\right).$$

We have

$$\lim_{t\to 0^+} t^{1-p} = \begin{cases} 0 & \text{if } 0 < 1-p, \\ \infty & \text{if } 1-p < 0 \end{cases}$$

and so we have

$$\int_0^1 \frac{\mathrm{d}x}{x^p} = \begin{cases} \frac{1}{1-p} & \text{if } p < 1 \\ \infty & \text{if } 1 \leq p. \end{cases}$$

**Remark 1.1.** Notice that this is almost exactly the inequality involving $p$ that we had for the Improper Integral of Type I

$$\int_1^\infty \frac{\mathrm{d}x}{x^p}$$

flipped the other way around. The remainder of this note will be devoted to a geometric explanation of *why* this occurs.

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## 2. Recollections on Composition Inverses

Now that we know for what values of $p$ both types of integrals converge, we can make some interesting geometric observations. In order to do so, we first we recall a few facts about inverse functions.

**Definition 2.1.** Let $f$ be a function. We say that $f$ has a **composition inverse** if there exists a function $f^{-1}$ such that for all $x$ in the domain of $f$

$$f^{-1} \circ f(x) = f^{-1}\left(f(x)\right) = x$$

and

$$f \circ f^{-1}(x) = f\left(f^{-1}(x)\right) = x.$$

We call $f^{-1}$ the **inverse of** $f$.

When such a function exists, the domain of $f$ is the range of $f^{-1}$ and the range of $f$ is the domain of $f^{-1}$.

An easy way to check if a function has an inverse is the following Theorem from.

**Theorem 2.2** (Horizontal Line Test)**.** *A function $f$ has a composition inverse if and only if any horizontal line intersects the graph of $f$ in at most one point.*

Geometrically, we also have a nice characterization of the graph of an inverse function. If $f$ admits a composition inverse, $f^{-1}$, then the graph of $y = f^{-1}(x)$ is the reflection of the graph of $y = f(x)$ across the line $y = x$.

## 3. Some Symmetry

We recall that we can compute the Improper Integral of Type I

$$\int_1^\infty \frac{\mathrm{d}x}{x^p}$$

as follows. When $p = 1$ we have

$$\int_1^\infty \frac{\mathrm{d}x}{x} = \lim_{t\to\infty} \int_1^t \frac{\mathrm{d}x}{x} = \lim_{t\to\infty} \ln(x)\Big|_1^t = \lim_{t\to\infty}(\ln(t) - \ln(1)) = \lim_{t\to\infty} \ln(t) = \infty$$

and when $p \neq 1$ we have

$$\int_1^\infty \frac{\mathrm{d}x}{x^p} = \lim_{t\to\infty} \int_1^t \frac{\mathrm{d}x}{x^p} = \lim_{t\to\infty} \int_1^t x^{-p}\,\mathrm{d}x = \lim_{t\to\infty} \frac{x^{1-p}}{1-p}\Big|_1^t = \lim_{t\to\infty} \left(-\frac{1}{p-1}\right)(t^{1-p} - 1).$$

We see that

$$\lim_{t\to\infty} t^{1-p} = \begin{cases} \infty & \text{if } 0 < 1-p, \\ 0 & \text{if } 1-p < 0 \end{cases}$$

so

$$\int_1^\infty \frac{\mathrm{d}x}{x^p} = \begin{cases} \infty & \text{if } p < 1, \\ \frac{1}{p-1} & \text{if } 1 < p. \end{cases}$$

Now fix a number $1 < p$, and consider the function $f(x) = x^{-p}$ defined on $(0, \infty)$. One can check that any function of this type passes the horizontal line test on $(0, \infty)$ or, algebraically, we can see the function $f^{-1}(x) = x^{-\frac{1}{p}}$ defined on $(0, \infty)$ satisfies

$$f \circ f^{-1}(x) = f\left(x^{-\frac{1}{p}}\right) = \left(x^{-\frac{1}{p}}\right)^{-p} = x^{\frac{p}{p}} = x$$

and

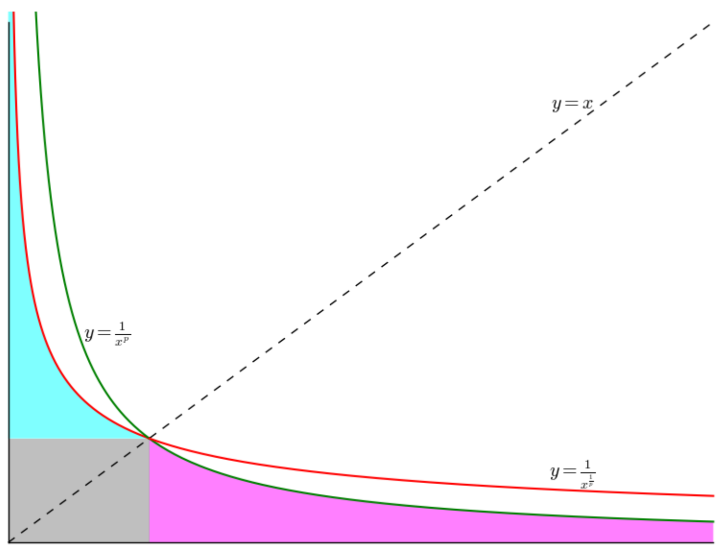
$$f^{-1} \circ f(x) = f^{-1}\left(x^{-p}\right) = \left(x^{-p}\right)^{-\frac{1}{p}} = x^{\frac{p}{p}} = x.$$

We make the observation that since $1 < p$ we also have $1/p < 1$, so we know both integrals

$$\int_0^1 f^{-1}(x)\,\mathrm{d}x = \int_0^1 \frac{\mathrm{d}x}{x^{\frac{1}{p}}} \text{ and } \int_1^\infty f(x)\,\mathrm{d}x = \int_1^\infty \frac{\mathrm{d}x}{x^p}$$

converge.

Moreover, we know that the graph of $f^{-1}(x)$ is the reflection of the graph of $f(x)$ along the line $y = x$, so we can see from the symmetry in the graph



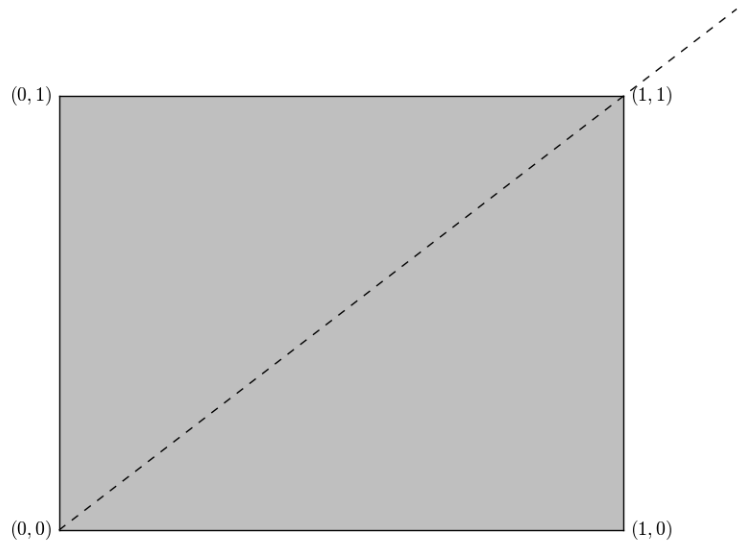

that the blue region and the pink region have the same area, which is given by

$$\int_1^\infty f(x)\,\mathrm{d}x = \int_1^\infty \frac{\mathrm{d}x}{x^p}.$$

By adding together the area grey region and the area of the blue region, we should obtain

$$\int_0^1 f^{-1}(x)\,\mathrm{d}x = \int_0^1 \frac{\mathrm{d}x}{x^{\frac{1}{p}}}$$

The grey region is just the square




which has area one, so one should reasonably expect that

$$\int_0^1 f^{-1}(x)\,\mathrm{d}x = 1 + \int_1^\infty f(x)\,\mathrm{d}x = 1 + \int_1^\infty \frac{\mathrm{d}x}{x^p} = 1 + \frac{1}{p-1} = \frac{p-1+1}{p-1} = \frac{p}{p-1}.$$

In fact, by our formula above, this is indeed the case:

$$\int_0^1 f^{-1}(x)\,\mathrm{d}x = \int_0^1 \frac{\mathrm{d}x}{x^{\frac{1}{p}}} = \frac{1}{1-\frac{1}{p}} = \frac{p}{p\left(1-\frac{1}{p}\right)} = \frac{p}{p-1}.$$

Since every number in the interval $(0,1)$ can be obtained as $1/p$ for some $1 < p$, this gives an intuitive geometric explaination for why the Improper Integral of Type II converges for $p < 1$ and diverges for $1 \leq p$, while the Improper Integral of Type I converges for $1 < p$ and diverges for $p \leq 1$.