E.g.: Find

$$\int \frac{dx}{x(x^2+1)^2}$$

2/1/18 1

$$\frac{1}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$$

$$\Rightarrow 1 = A(x^2+1)^2 + (Bx+C)x(x^2+1) + (Dx+E)x$$

$$= A(x^4+2x^2+1) + (Bx^2+Cx)(x^2+1) + Dx^2+Ex$$

$$= A(x^4+2x^2+1) + Bx^4+Bx^2+Cx^3+Cx+Dx^2+Ex$$

$$= (A+B)x^4 + Cx^3 + (2A+B+D)x^2 + (C+E)x + A$$

$0 = A+B \Rightarrow -A = +B = -1$

$0 = C.$

$0 = 2A+B+D \Rightarrow D = -B-2A = 1-2 = -1$

$0 = C+E \Rightarrow E = 0$

$1 = A$

$A=1,\ B=-1,\ C=0,\ D=-1,\ E=0$

$$\int \frac{dx}{x(x^2+1)^2} = \int \frac{dx}{x} + \int \frac{-x}{x^2+1}dx + \int \frac{-x}{(x^2+1)^2}dx$$

$$= \ln|x| - \frac{1}{2}\int \frac{du}{u} - \frac{1}{2}\int \frac{du}{u^2}$$

$$= \ln|x| - \frac{1}{2}\ln(x^2+1) - \frac{1}{2}(-1)u^{-1} + C$$

$$= \ln|x| - \frac{1}{2}\ln(x^2+1) + \frac{1}{2(x^2+1)} + C.$$

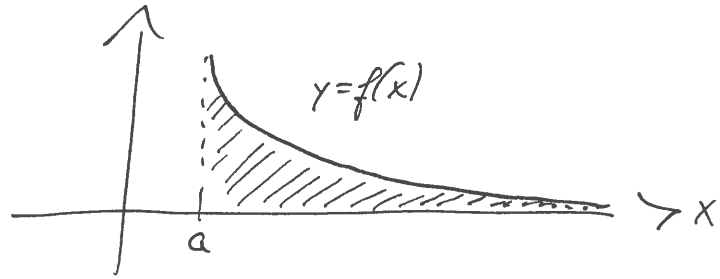
$u = x^2+1$

$du = 2x\,dx$

$-\frac{1}{2}du = -x\,dx$

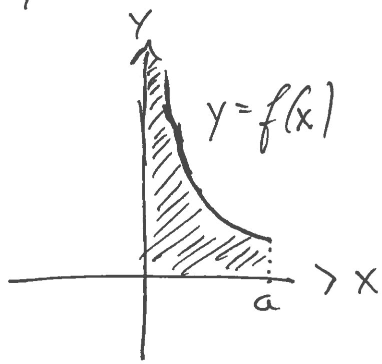
8.8 : Improper Integrals

What about integrals on ~~infiti-~~ infinite regions

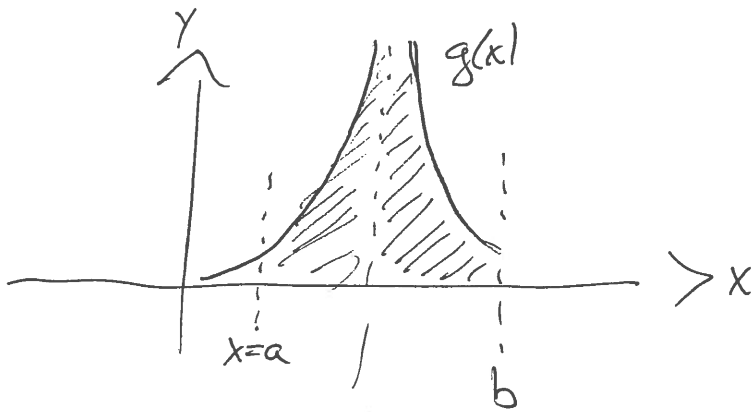


$$\int_a^\infty f(x)\,dx = ?$$

or unbounded/discontinuous functions on a finite interval?



$$\int_0^a f(x)\,dx = ?$$



$$\int_a^b g(x)\,dx = ?$$

Idea: Say $f(x)$ is continuous on $[a,\infty)$, by the FTC (3)
we have a continuous function

$$F(x) = \int_a^x f(t)\,dt$$

The area of the orange region is the value of $F(x)$. This is a continuous function. Asking "what is $\int_a^\infty f(x)\,dx$" is the same as asking "does

$$F(x) = \int_a^x f(t)\,dt$$

have a horizontal asymptote?"

We define for f continuous on $[a,\infty)$

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$$

We define (symmetrically) for f continuous on $(-\infty, a]$

$$\int_{-\infty}^a f(x)\,dx = \lim_{t\to\infty} \int_t^a f(x)\,dx$$

We define for f continuous, for c any real number

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx$$

If the limits are finite, then we say the <u>integral converges</u>, and this limit is the <u>value</u> of the integral. Otherwise, we say the <u>integral</u> <u>diverges</u>.

E.g.: $f(x) = \frac{\ln(x)}{x}$ on $[1,\infty)$ Is the area under f finite?

For any value t, take $u = \ln(x)$, $du = \frac{dx}{x}$

$$\int_1^t \frac{\ln(x)}{x}\,dx = \int_0^{\ln(t)} u\,du = \frac{1}{2}u^2\Big|_0^{\ln(t)} = \frac{1}{2}\left(\ln(t)^2 - 0\right) = \frac{1}{2}\ln(t)^2$$

$$\int_1^\infty \frac{\ln(x)}{x}\,dx = \lim_{t\to\infty}\int_1^t \frac{\ln(x)}{x}\,dx = \lim_{t\to\infty}\frac{1}{2}\ln(t)^2 = \infty.$$

E.g.: Take $p>0$, then $\int_1^\infty \frac{\ln(x)}{x^{1+p}}\,dx$ is finite!

$u = \ln(x)$ $\quad v = \frac{-1}{px^p}$

$du = \frac{dx}{x}$ $\quad dv = \frac{1}{x^{1+p}}$

$$\int_1^t \frac{\ln(x)}{x^{1+p}}\,dx = \frac{-1}{px^p}\ln(x)\Big|_1^t + \frac{1}{p}\int_1^t \frac{dx}{x^{p+1}} = \frac{-1}{px^p}\ln(x)\Big|_1^t + \frac{1}{p}\left(\frac{-1}{px^p}\right)\Big|_1^t$$

$$= \frac{-1}{p}\left(\frac{1}{t^p}\ln(t) - 0\right) - \frac{1}{p^2}\left(\frac{1}{t^p} - 1\right)$$

$$= -\frac{1}{p}\left(\frac{1}{t^p}\right)\ln(t) - \frac{1}{p^2}\left(\frac{1}{t^p}\right) + \frac{1}{p^2}$$

$$\int_1^\infty \frac{\ln(x)}{x^{1+p}}\,dx = \lim_{t\to\infty}\int_1^t \frac{\ln(x)}{x^{1+p}}\,dx = \lim_{t\to\infty}\left(-\frac{1}{p}\underline{\left(\frac{1}{t^p}\right)\ln(t)} - \underline{\frac{1}{p^2}\left(\frac{1}{t^p}\right)} + \frac{1}{p^2}\right)$$

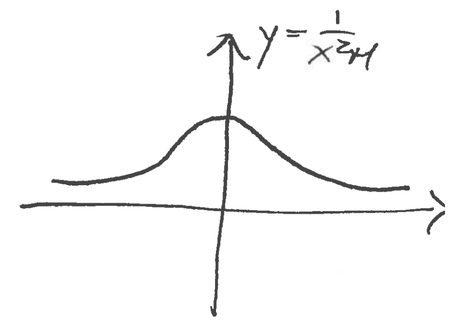
$$\lim_{t\to\infty} \frac{\ln(t)}{t^p} \overset{\text{L'Hôpital}}{=} \lim_{t\to\infty} \frac{\left(\frac{1}{t}\right)}{pt^{p-1}} = \lim_{t\to\infty} \frac{1}{pt^p} = 0.$$

(this is where we use $p>0$)

$$\lim_{t\to\infty} \frac{1}{t^p} = 0, \quad \lim_{t\to\infty} \frac{1}{p^2} = \frac{1}{p^2}.$$

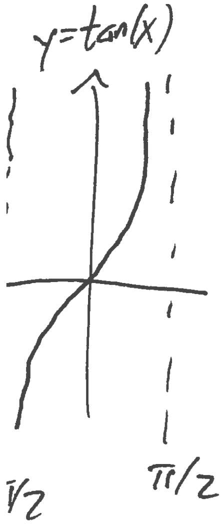
$$\Rightarrow \int_1^\infty \frac{\ln(x)}{x^{1+p}}\,dx = \frac{1}{p}\lim_{t\to\infty}\frac{\ln(t)}{t^p} - \frac{1}{p^2}\lim_{t\to\infty}\frac{1}{t^p} + \lim_{t\to\infty}\frac{1}{p^2}$$

$$= \frac{1}{p}(0) - \frac{1}{p^2}(0) + \frac{1}{p^2} = \frac{1}{p^2}!$$



E.g:

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \lim_{t\to-\infty}\int_t^0 \frac{dx}{1+x^2} + \lim_{t\to\infty}\int_0^t \frac{dx}{1+x^2}$$



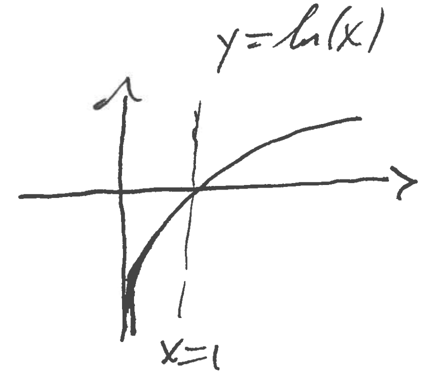
$$= \lim_{t\to-\infty} \arctan(x)\Big|_t^0 + \lim_{t\to\infty} \arctan(x)\Big|_0^t$$

$$= \lim_{t\to-\infty}\left(\arctan(0) - \arctan(t)\right) + \lim_{t\to\infty}\left(\arctan(t) - \arctan(0)\right)$$

$$= \lim_{t\to-\infty} -\arctan(t) + \lim_{t\to\infty} \arctan(t)$$

$$= -\left(\frac{-\pi}{2}\right) + \pi/2 = 2(\pi/2) = \pi.$$

Eg: $\displaystyle\int_1^\infty \frac{dx}{x^p} = \begin{cases} \infty & \text{if } p \le 1, \\ \frac{1}{p-1} & \text{if } p > 1. \end{cases}$



$p = 1$

$$\int_1^\infty \frac{dx}{x} = \lim_{t \to \infty} \int_1^t \frac{dx}{x} = \lim_{t\to\infty} \ln(t) = \infty.$$

$p \neq 1$

$$\int_1^\infty \frac{dx}{x^p} = \lim_{t\to\infty} \int_1^t x^{-p}\,dx = \lim_{t\to\infty} \left. \frac{x^{1-p}}{1-p} \right|_1^t$$

$$= \lim_{t\to\infty} \left(\frac{t^{1-p}}{1-p} - \frac{1}{1-p} \right)$$

If $p > 1$, then $\underset{0 < p-1}{\underset{\Updownarrow}{1-p < 0}}$, so $t^{1-p} = \frac{1}{t^{p-1}} \to 0$ as $t \to \infty$.

So

$$\int_1^\infty \frac{dx}{x^p} = \lim_{t\to\infty} \left(\frac{1}{(1-p)t^{p-1}} - \frac{1}{1-p} \right) = \lim_{t\to\infty} \left(\frac{1}{1-p} \frac{1}{t^{p-1}} \right) - \lim_{t\to\infty} \frac{1}{1-p}$$

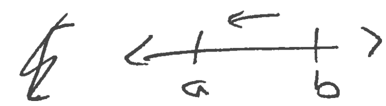
$$= 0 - \frac{1}{1-p} = \boxed{\frac{1}{p-1} \text{ when } p > 1}$$

If $p < 1$, $\lim_{t\to\infty} t^{1-p} = \infty$, so $\int_1^\infty \frac{dx}{x^p} = \infty$.

Type II

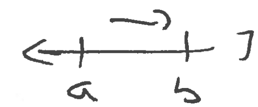
1) f is continuous on $(a,b]$, discont. at a

$$\int_a^b f(x)\,dx = \lim_{t \to a^+} \int_t^b f(x)\,dx$$



2) f is continuous on $[a,b)$, discont. at b

$$\int_a^b f(x)\,dx = \lim_{t \to b^-} \int_a^t f(x)\,dx$$



3) f is continuous on $[a,c) \cup (c,b]$, $a < c < b$, discont. at c

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Eg.: Evaluate

$$\int_0^3 \frac{dx}{(x-1)^{2/3}} = \int_0^1 \frac{dx}{(x-1)^{2/3}} + \int_1^3 \frac{dx}{(x-1)^{2/3}}$$

$u = x-1$

$du = dx$

$$= \int_{-1}^0 \frac{du}{u^{2/3}} + \int_0^2 \frac{du}{u^{2/3}}$$

$$\int_{-1}^0 \frac{du}{u^{2/3}} = \lim_{t\to 0^-} \int_{-1}^t \frac{du}{u^{2/3}} = \lim_{t\to 0^-} 3u^{1/3}\Big|_{-1}^t = 0 - 3(-1)^{1/3} = 3.$$

$$\int_0^2 \frac{du}{u^{2/3}} = \lim_{t\to 0^+} 3u^{1/3}\Big|_t^2 = 3(2)^{1/3} - 0$$

$$\Rightarrow \int_0^3 \frac{dx}{(x-1)^{2/3}} = 3 + 3\sqrt[3]{2} = 3(1+\sqrt[3]{2}).$$