8.5 : Partial Fraction Decomposition

1/30/17

Idea: Recall, a polynomial is a function

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

a_i's are numbers, and a rational function is one of the form

$$r(x) = \frac{p(x)}{q(x)}, \quad p, q \text{ are polynomials.}$$

We want to integrate rational functions. Some we can deal with:

$$\int \frac{dx}{bx+c} = \frac{1}{b}\int \frac{du}{u} = \frac{1}{b}\ln|u| + d = \frac{\ln|bx+c|}{b} + d.$$

$u = bx+c$

$du = b\,dx$

$\frac{1}{b}du = dx$

$$\int \frac{ax+b/2}{ax^2+bx+c}\,dx = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|ax^2+bx+c| + d.$$

$u = ax^2+bx+c$

$du = 2ax+b$

$\frac{1}{2}du = ax+b/2$

Motivational Example

$$\int \frac{5x-3}{x^2-2x-3}\,dx$$

Factor $x^2-2x-3 = (x-3)(x+1)$

Observe: $\dfrac{2}{x+1} + \dfrac{3}{x-3} = \dfrac{2(x-3)+3(x+1)}{x^2-2x-3}$

$$= \frac{2x-6+3x+3}{x^2-2x-3}$$

$$= \frac{5x-3}{x^2-2x-3}$$

So

$$\int \frac{5x-3}{x^2-2x-3}\,dx = \int\left(\frac{2}{x+1}+\frac{3}{x-3}\right)dx = 2\int\frac{dx}{x+1} + 3\int\frac{dx}{x-3}$$

$$= 2\ln|x+1| + 3\ln|x-3| + C.$$

Fun Fact: There are no irreducible polynomials of degree at least 3

Recall: degree of $p(x) = a_n x^n + \cdots + a_1 x + a_0$ is n.

a degree 2 polynomial is irreducible if it has no roots

$$ax^2+bx+c, \quad D = b^2-4ac$$

has no roots if and only if $D < 0$.

Every polynomial with real coefficients can be written as a product of linear factors (i.e. polynomials of the form $ax+b$) and irreducible quadratics (ax^2+bx+c, $b^2-4ac<0$).

General Method: $r(x) = \frac{f(x)}{g(x)}$.

0 Factor $g(x)$ into a product of linear terms & irreducible quadratics

1 For each linear factor $(x-s)$, suppose that $(x-s)^m$ is the highest power of $(x-s)$ appearing in the factorization of g. Then assign to this factor

$$\frac{A_1}{(x-s)} + \frac{A_2}{(x-s)^2} + \cdots + \frac{A_m}{(x-s)^m}.$$

2 For each irreducible quadratic, x^2+px+q, assume that $(x^2+px+q)^n$ is the highest power in the factorization. Then assign to this factor

$$\frac{B_1x+C_1}{x^2+px+q} + \frac{B_2x+C_2}{(x^2+px+q)^2} + \cdots + \frac{B_nx+C_n}{(x^2+px+q)^2}.$$

1 Add up the result of steps 2 and 3, set it equal $r(x)$, clear denominators, equate coefficients, solve for the A's, B's, & C's.

E.g: Use partial fractions to evaluate

$$\int \frac{x^2+4x+1}{x^3+3x^2-x-3}\,dx.$$

To factor x^3+3x^2-x-3, find a root.

Try 1: $1^3+3(1)^2-1-3 = \cancel{1}+\cancel{3}-\cancel{1}-\cancel{3} = 0$. This says that

$(x-1)$ is a factor of x^3+3x^2-x-3.

To figure out what the degree 2 factor is, do polynomial long division.

$$\begin{array}{r|l} & x^2+4x+3 \\ \hline (x-1) & x^3+3x^2-x-3 \\ & \underline{-x^3+x^2} \\ & \quad 4x^2-x \\ & \quad \underline{-4x^2+4x} \\ & \qquad 3x-3 = 3(x-1) \\ & \qquad \underline{-3x+3} \\ & \qquad\quad 0 \end{array}$$

$$x^3+3x^2-x-3 = (x-1)(x^2+4x+3)$$

$$= (x-1)(x+1)(x+3)$$

$$\int \frac{x^2+4x+1}{x^3+3x^2-x-3}\,dx = \int \frac{x^2+4x+1}{(x-1)(x+1)(x+3)}\,dx$$

$$\frac{x^2+4x+1}{x^3+3x^2-x-3} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{x+3}$$

Multiply both sides by $(x-1)(x+1)(x+3)$ to get

$$x^2+4x+1 = A(x+1)(x+3) + B(x-1)(x+3) + C(x-1)(x+1)$$

$$= A(x^2+4x+3) + B(x^2+2x-3) + C(x^2-1)$$

$$= x^2(A+B+C) + x(4A+2B) + (3A-3B-C).$$

$1 = A+B+C$

$4 = 4A+2B \iff 2 = 2A+B$

$1 = 3A-3B-C$

Solve: $B = 2-2A$, substitute this into the equations

$1 = A+(2-2A)+C = -A+C+2 \iff -1 = -A+C$

$1 = 3A-3(2-2A)-C = 3A-6+6A-C = 9A-C-6 \iff 7 = 9A-C$

Use the first equation to get $C = A-1$. Second equation

$7 = 9A-(A-1) = 8A+1 \iff 8A = 6 \iff \boxed{A = 3/4.}$

$B = 2 - 2A = 2 - 2\left(\frac{3}{4}\right) = \frac{4}{2} - \frac{3}{2} = \frac{1}{2}.$

$C = A - 1 = \frac{3}{4} - \frac{4}{4} = \frac{-1}{4}.$

$$\int \frac{x^2+4x+1}{x^3+3x^2-x-3}\,dx = \int \frac{A}{x-1}\,dx + \int \frac{B}{x+1}\,dx + \int \frac{C}{x+3}\,dx$$

$$= \frac{3}{4}\int \frac{dx}{x-1} + \frac{1}{2}\int \frac{dx}{x+1} + \left(\frac{-1}{4}\right)\int \frac{dx}{x+3}$$

$$= \frac{3}{4}\ln|x-1| + \frac{1}{2}\ln|x+1| - \frac{1}{4}\ln|x+3| + C.$$

E.g.: Use partial fractions to evaluate

$$\int \frac{6x+7}{(x+2)^2}\,dx$$

$$\frac{6x+7}{(x+2)^2} = \frac{A}{x+2} + \frac{B}{(x+2)^2} \Rightarrow 6x+7 = A(x+2) + B. = Ax + (2A+B)$$

$6 = A$

$7 = 2A + B = 12 + B \Rightarrow B = 7 - 12 = -5$

$$\int \frac{6x+7}{(x+2)^2}\,dx = 6\int \frac{dx}{x+2} + (-5)\int \frac{dx}{(x+2)^2} = 6\ln|x+2| - 5(-1)(x+2)^{-1} + C$$

$$= 6\ln|x+2| + \frac{5}{x+2} + C.$$

Rmk: If the degree of the numerator is larger than the degree of the denominator, you must first do poly long division

Eg: $\int \frac{2x^3-4x^2-x-3}{x^2-2x-3}\,dx$

$$\begin{array}{r} 2x \\ x^2-2x-3 \,\big)\, 2x^3-4x^2-x-3 \\ -2x^3+4x^2+6x \\ \hline 5x-3 \end{array}$$

$\Rightarrow 2x^3-4x^2-x-3 = 2x(x^2-2x-3)+5x-3$

$\Rightarrow \frac{2x^3-4x^2-x-3}{x^2-2x-3} = \boxed{2x + \frac{5x-3}{x^2-2x-3}}$

$$\int \frac{2x^3-4x^2-x-3}{x^2-2x-3}\,dx = \int 2x\,dx + \int \frac{5x-3}{x^2-2x-3}\,dx$$

$x^2-2x-3=(x-3)(x+1)$

$$\frac{5x-3}{x^2-2x-3} = \frac{A}{x-3} + \frac{B}{x+1} \Rightarrow 5x-3 = A(x+1)+B(x-3)$$
$$= (A+B)x + A-3B$$

$A+B=5$
$A-3B=-3$

Solve: $B=2,\ A=3$

$$\int \frac{5x-3}{x^2-2x-3}\,dx = 3\int \frac{dx}{x-3} + 2\int \frac{dx}{x+1} = 3\ln|x-3| + 2\ln|x+1| + C$$

$$\int \frac{2x^3-4x^2-x-3}{x^2-2x-3}\,dx = \int 2x\,dx + 3\ln|x-3| + 2\ln|x+1| = x^2 + 3\ln|x-3| + 2\ln|x+1| + C$$