8.4 : Trig. Substitution

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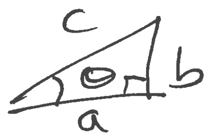
Recall:



Pythagoras: $a^2+b^2=c^2$

$a=\sqrt{c^2-b^2}$

$b=\sqrt{c^2-a^2}$

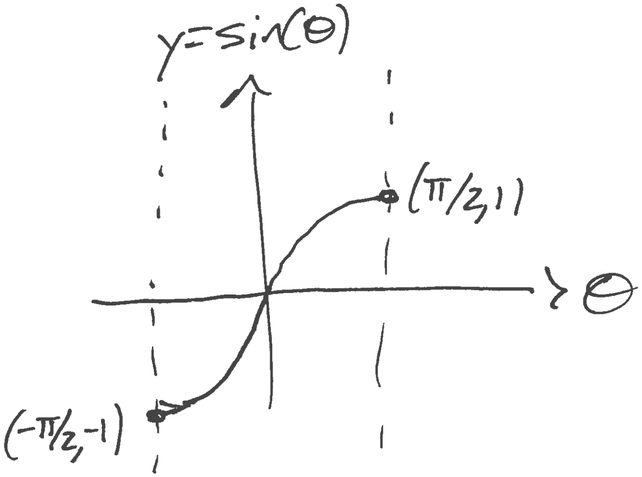
$c=\sqrt{a^2+b^2}$

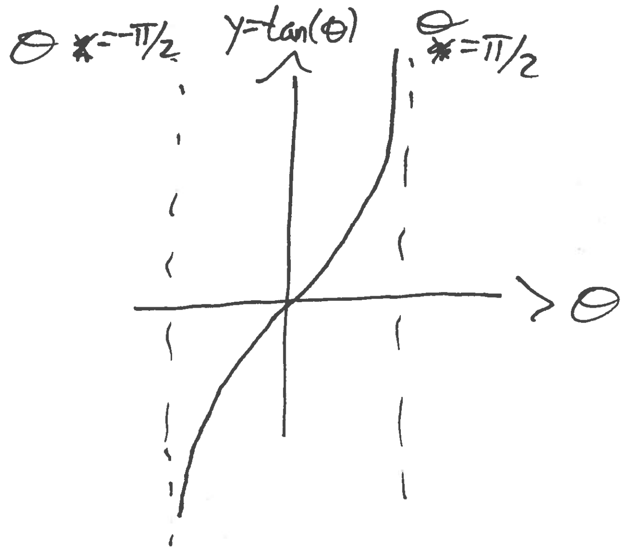
$\sin(\theta)=b/c$

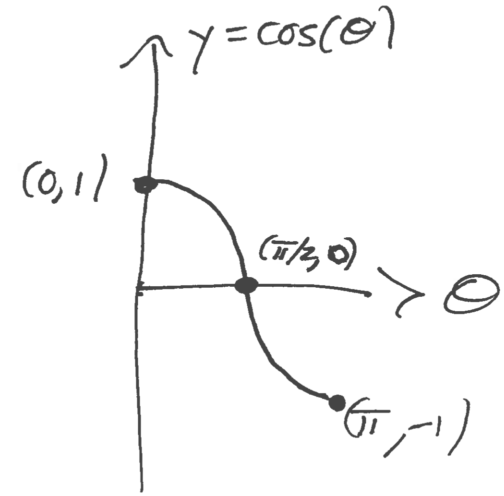
$\cos(\theta)=a/c$

$\tan(\theta)=b/a$

$\sec(\theta)=\frac{1}{\cos(\theta)}=\frac{c}{a}$



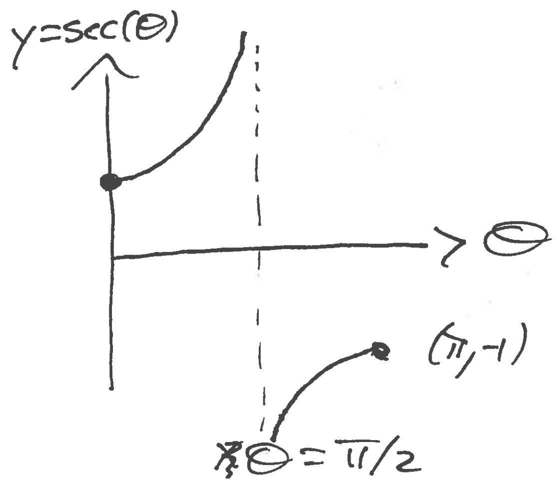




$\sin(\theta)$ defined on $[-\pi/2, \pi/2]$

$\tan(\theta)$ defined on $(-\pi/2, \pi/2)$

$\sec(\theta)$ defined on $[0, \pi/2) \cup (\pi/2, \pi]$



We want to exploit the trig identities

$$1+\tan^2(\theta)=\sec^2(\theta),$$
$$1-\sin^2(\theta)=\cos^2(\theta), \text{ and}$$
$$\sec^2(\theta)-1=\tan^2(\theta)$$

to compute integrals involving

① $\sqrt{a^2+x^2}$, ② $\sqrt{a^2-x^2}$, ③ $\sqrt{x^2-a^2}$

① Let $x=a\tan(\theta)$, so

$$\sqrt{a^2+x^2}=\sqrt{a^2+(a\tan(\theta))^2}=\sqrt{a^2(1+\tan^2(\theta))}=\sqrt{a^2\sec^2(\theta)}=a\sec(\theta).$$

② Let $x=a\sin(\theta)$

$$\sqrt{a^2-x^2}=\sqrt{a^2-a^2\sin^2(\theta)}=\sqrt{a^2\cos^2(\theta)}=a\cos(\theta)$$

③ Let $x=a\sec(\theta)$

$$\sqrt{x^2-a^2}=\sqrt{a^2\sec^2(\theta)-a^2}=\sqrt{a^2\tan^2(\theta)}=a\tan(\theta).$$

E.g.: Evaluate

$$\int \frac{dx}{\sqrt{4+x^2}} = \int \frac{dx}{\sqrt{2^2+x^2}}$$

$a=2$, let $x = a\tan(\theta) = 2\tan(\theta)$

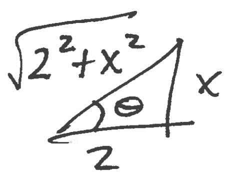
$$\sqrt{2^2+x^2} = \sqrt{2^2+2^2\tan^2(\theta)} = \sqrt{4(1+\tan^2(\theta))} = \sqrt{4}\sqrt{\sec^2(\theta)} = 2\sec(\theta)$$

$\underline{dx = 2\sec^2(\theta)\,d\theta}$ $\left[\text{Formally } \frac{dx}{d\theta} = \frac{d}{d\theta}(2\tan(\theta)) = \frac{dx}{d\theta} = 2\sec^2(\theta) \right.$

$\left. \Rightarrow dx = 2\sec^2(\theta)\,d\theta \right]$

$$\int \frac{dx}{\sqrt{4+x^2}} = \int \frac{2\sec^2(\theta)}{2\sec(\theta)}\,d\theta = \int \sec(\theta)\,d\theta = \ln\left|\sec(\theta) + \underline{\tan(\theta)}\right| + C.$$

$$= \ln\left|\frac{\sqrt{4+x^2}}{2} + \frac{x}{2}\right| + C.$$



$\frac{x}{2} = \tan(\theta)$

$\sec(\theta) = \frac{1}{\cos(\theta)} = \frac{\sqrt{4+x^2}}{2}.$

E.g.: Evaluate $\int \frac{x^2}{\sqrt{9-x^2}}\,dx$

$$\boxed{\begin{aligned}\sin^2(\theta) &= \frac{1-\cos(2\theta)}{2}\\ \cos^2(\theta) &= \frac{1+\cos(2\theta)}{2}\end{aligned}}$$

4

$$x = a\sin(\theta) = 3\sin(\theta)$$

$$\sqrt{9-x^2} = \sqrt{9-9\sin^2(\theta)} = \sqrt{9\cos^2(\theta)} = 3\cos(\theta).$$

$$\frac{dx}{d\theta} = 3\cos(\theta) \Rightarrow dx = 3\cos(\theta)\,d\theta.$$

$$\int \frac{x^2}{\sqrt{9-x^2}}\,dx = \int \frac{3^2\sin^2(\theta)}{\cancel{3\cos(\theta)}}\,\cancel{3\cos(\theta)}\,d\theta = 9\int \sin^2(\theta)\,d\theta.$$

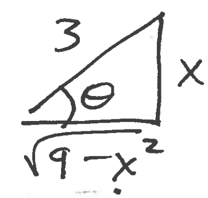
$$= 9\int \frac{1-\cos(2\theta)}{2}\,d\theta = \frac{9}{2}\int d\theta - \frac{9}{2}\int \cos(2\theta)\,d\theta$$

This is where we use $\sin(\theta)$ defined for $-\pi/2 \le \theta \le \pi/2$

$$= \frac{9}{2}\theta - \frac{9}{2}\left(\frac{1}{2}\right)\sin(2\theta) + C$$

$$= \frac{9}{2}\theta - \frac{9}{4}\sin(2\theta) + C.$$

$$= \frac{9}{2}\theta - \frac{9}{4}2\sin(\theta)\cos(\theta) + C.$$



$$\sin(\theta) = \frac{x}{3}$$

$$\arcsin(\sin(\theta)) = \arcsin\left(\frac{x}{3}\right)$$

$$\parallel$$

$$\theta$$

$$= \frac{9}{2}\arcsin\left(\frac{x}{3}\right) - \frac{\cancel{9}}{2}\left(\frac{x}{\cancel{3}}\right)\left(\frac{\sqrt{9-x^2}}{\cancel{3}}\right) + C = \frac{9}{2}\arcsin\left(\frac{x}{3}\right) - \frac{x\sqrt{9-x^2}}{2} + C.$$

E.g: Evaluate $\int \frac{dx}{\sqrt{25x^2-4}}$, $x > 2/5$.

$$25x^2 - 4 = 5^2x^2 - 2^2 = 5^2\left(x^2 - \frac{2^2}{5^2}\right) = 5^2\left(x^2 - \left(\frac{2}{5}\right)^2\right)$$

$$\sqrt{25x^2-4} = \sqrt{5^2\left(x^2 - (2/5)^2\right)} = 5\sqrt{x^2 - (2/5)^2}$$

Let $x = a\sec(\theta) = 2/5 \sec(\theta)$ Rmk: we've tacitly assumed that $0 < \theta < \pi/2$ because $x > 0$, and $\sec(\theta) < 0$ when $\pi/2 < \theta \leq \pi$.

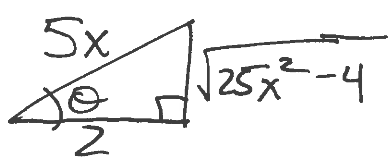
$$\sqrt{25x^2-4} = 5\sqrt{(2/5\sec(\theta))^2 - (2/5)^2} = 5\sqrt{(2/5)^2(\sec^2(\theta) - 1)}$$

$$= 5\left(\frac{2}{5}\right)\sqrt{\tan^2(\theta)} = 2\tan(\theta).$$

$$\frac{dx}{d\theta} = 2/5\sec(\theta)\tan(\theta) \Rightarrow dx = 2/5\sec(\theta)\tan(\theta)\,d\theta.$$

$$\int \frac{dx}{\sqrt{25x^2-4}} = \int \frac{\cancel{2}/5\sec(\theta)\cancel{\tan(\theta)}\,d\theta}{\cancel{2}\cancel{\tan(\theta)}} = \frac{1}{5}\int \sec(\theta)\,d\theta = 1/5 \ln\left|\sec(\theta) + \tan(\theta)\right| + C.$$

$\sec(\theta) = 5/2\, x = \frac{5x}{2}$



$\tan(\theta) = \frac{\sqrt{25x^2-4}}{2}$

$$= \frac{1}{5}\ln\left|\frac{5x}{2} + \frac{\sqrt{25x^2-4}}{2}\right| + C$$

$$= 1/5 \ln\left(\frac{5x}{2} + \frac{\sqrt{25x^2-4}}{2}\right) + C$$

because $x > 2/5$, $\sqrt{25x^2-4}$ is positive.

Eg.: One can compute arcsinh(x) with these techniques. ⑥

$$\sinh(u) = \frac{e^u - e^{-u}}{2}, \quad \cosh(u) = \frac{e^u + e^{-u}}{2}$$

Rmk: Euler's formula

$$\sin(u) = \frac{e^{iu} - e^{-iu}}{2i}, \quad \cos(u) = \frac{e^{iu} + e^{-iu}}{2}$$

There's a relation

$$-\sinh^2(u) + \cosh^2(u) = 1$$

Let $x = a\sinh(u)$, $dx = a\cosh(u)\,du$

$$\int \frac{dx}{\sqrt{a^2+x^2}} = \int \frac{a\cosh(u)\,du}{\sqrt{a^2 + a^2\sinh^2(u)}} = \int \frac{a\cosh(u)\,du}{\sqrt{a^2\cosh^2(u)}} = \int \frac{a\cosh(u)}{a\cosh(u)}\,du = \int du$$

$$= u + C$$

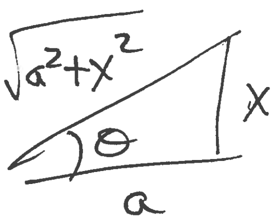
$$x = a\sinh(u) \Rightarrow \frac{x}{a} = \sinh(u) \Rightarrow u = \operatorname{arcsinh}\left(\frac{x}{a}\right)$$

$$\int \frac{dx}{\sqrt{a^2+x^2}} = \operatorname{arcsinh}\left(\frac{x}{a}\right) + C.$$

Let $x = a\tan(\theta)$, $dx = a\sec^2(\theta)\,d\theta$

$$\sqrt{a^2 + a^2\tan^2(\theta)} = \sqrt{a^2(1+\tan^2(\theta))} = a\sqrt{\sec^2(\theta)} = a\sec(\theta).$$

$$\int \frac{dx}{\sqrt{a^2+x^2}} = \int \frac{a\sec^2(\theta)\,d\theta}{a\sec(\theta)} = \int \sec(\theta)\,d\theta = \ln|\sec(\theta) + \tan(\theta)| + C$$



$\tan(\theta) = \frac{x}{a}$

$\sec(\theta) = \frac{\sqrt{a^2+x^2}}{a}$

$$= \ln\left|\frac{\sqrt{a^2+x^2}}{a} + \cancel{\tan(\theta)}\,\frac{x}{a}\right| + C.$$

$$\frac{e^u - e^{-u}}{2} = 0 \iff e^u - e^{-u} = 0$$

$$\iff e^u = e^{-u}$$

$$\iff e^{2u} = 1$$

$$\iff u = 0$$

$\sinh(0) = 0$

$\Rightarrow \operatorname{arcsinh}(\underset{0}{\underset{\|}{\sinh(0)}}) = 0$

$\ln\left|\frac{\sqrt{a^2+x^2}}{a} + \frac{x}{a}\right|$ and $\operatorname{arcsinh}\left(\frac{x}{a}\right)$

differ only by a constant

$$\ln\left|\frac{\sqrt{a^2+0}}{a} + \frac{0}{a}\right| = \ln\left|\frac{a}{a}\right| = \ln|1| = 0$$

$\Rightarrow$ this constant is zero

$$\Rightarrow \operatorname{arcsinh}(x) = \ln\left|\frac{\sqrt{a^2+x^2}}{a} + \frac{x}{a}\right|.$$