8.3: Trigonometric Integrals

Products of Sines & Cosines

Want to integrate something of the form $\sin^m(x)\cos^n(x)$, m, n are integers.

1) m is odd

Write $m = 2k+1$, for k some integer.

$$\sin^m(x) = \sin^{2k+1}(x) = \sin^{2k}(x)\sin(x) = (\sin^2(x))^k \sin(x)$$

Know $\sin^2(x) + \cos^2(x) = 1$, rewrite

$$\sin^m(x) = (1-\cos^2(x))^k \sin(x)$$

let $u = \cos(x)$, $du = -\sin(x)dx$, $-du = \sin(x)dx$

$$\int \sin^m(x)\cos^n(x)dx = \int (1-\cos^2(x))^k \sin(x)\cos^n(x)dx$$

$$= -\int (1-u^2)^k u^n du$$

2) m is even, n is odd.

Write $n = 2k+1$, for k some integer

$$\cos^n(x) = \cos^{2k+1}(x) = \cos^2(x)^k \cos(x) = (1-\sin^2(x))^k \cos(x)$$

$u = \sin(x)$
$du = \cos(x)dx$

$$\int \sin^m(x)\cos^n(x)dx = \int u^m (1-u^2)^k du$$

3) Both m & n even

Recall: $\sin^2(x) = \dfrac{1-\cos(2x)}{2}$, $\cos^2(x) = \dfrac{1+\cos(2x)}{2}$

Write $m = 2\ell$, $n = 2k$, ℓ, k integers

$$\int \sin^m(x)\cos^n(x)\,dx = \int \sin^{2\ell}(x)\cos^{2k}(x)\,dx$$

$$= \int (\sin^2(x))^\ell (\cos^2(x))^k\,dx$$

$$= \int \left(\frac{1-\cos(2x)}{2}\right)^\ell \left(\frac{1-\cos(2x)}{2}\right)^k dx$$

This gives an integrand involving smaller powers of cosine when expanded

E.g.: Evaluate $\int \sin^3(x)\cos^2(x)\,dx$: power of sine is odd, case 1.

$$\int \sin^3(x)\cos^2(x)\,dx = \int \sin^2(x)\cos^2(x)\sin(x)\,dx$$

$$= \int (1-\cos^2(x))\cos^2(x)\sin(x)\,dx \qquad u = \cos(x), \quad -du = \sin(x)dx$$

$$= -\int (1-u^2)u^2\,du$$

$$= -\int [u^2 - u^4]\,du = -\int u^2\,du + \int u^4\,du$$

$$= -\frac{1}{3}u^3 + \frac{1}{5}u^5 + C = \frac{-1}{3}\cos^3(x) + \frac{1}{5}\cos^5(x) + C.$$

E.g: Evaluate $\int \cos^5(x)dx$. $m=0$, $n=5$ case 2.

$$\int \cos^5(x)dx = \int \cos^4(x)\cos(x)dx$$

$$= \int (1-\sin^2(x))^2 \underbrace{\cos(x)dx}_{du}$$

$u = \sin(x)$

$du = \cos(x)dx$

$$= \int (1-u^2)^2 du$$

$$= \int (1-2u^2+(u^2)^2)du$$

$$= \int du - 2\int u^2 du + \int u^4 du$$

$$= u - 2\left(\tfrac{1}{3}\right)u^3 + \tfrac{1}{5}u^5 + C$$

$$= \sin(x) - 2/3\sin^3(x) + 1/5\sin^5(x) + C.$$

Eg: Evaluate $\int \sin^2(x)\cos^4(x)\,dx$ case 3

$$\int \sin^2(x)\cos^4(x)\,dx = \int \left(\frac{1-\cos(2x)}{2}\right)\left(\frac{1+\cos(2x)}{2}\right)^2 dx$$

$$= \frac{1}{8}\int \left[(1-\cos(2x))(1+\cos(2x))^2\right] dx$$

$$= \frac{1}{8}\int (1-\cos(2x))(1+2\cos(2x)+\cos^2(2x))]\,dx$$

$$= \frac{1}{8}\int [1+2\cos(2x)+\cos^2(2x)-\cos(2x)-2\cos^2(2x)-\cos^3(2x)]\,dx$$

$$= \frac{1}{8}\int [1+\cos(2x)-\cos^2(2x)-\cos^3(2x)]\,dx$$

$$= \frac{1}{8}\left(\int dx + \int \cos(2x)\,dx - \int \cos^2(2x)\,dx - \int \cos^3(2x)\,dx\right)$$

$$= \frac{1}{8}\left(\int dx + \int \cos(2x)\,dx - \int \frac{1+\cos(4x)}{2}\,dx - \int (1-\sin^2(2x))\cos(2x)\,dx\right)$$

$u = \sin(2x)$
$du = 2\cos(2x)\,dx$
$\frac{1}{2}du = \cos(2x)\,dx$

$$= \frac{1}{8}\left(\int dx + \int \cos(2x)\,dx - \frac{1}{2}\int dx - \frac{1}{2}\int \cos(4x)\,dx - \frac{1}{2}\int (1-u^2)\,du\right)$$

$$= \frac{1}{8}\left(\int dx + \int \cos(2x)\,dx - \frac{1}{2}\int \cos(4x)\,dx - \frac{1}{2}\int du + \frac{1}{2}\int u^2\,du\right)$$

$$= \frac{1}{8}\left(\frac{1}{2}\int dx + \int \cos(2x)\,dx - \frac{1}{2}\int \cos(4x)\,dx - \frac{1}{2}u + \frac{1}{6}u^3\right) + C$$

$$= \frac{1}{8}\left(\frac{1}{2}x + \frac{1}{2}\sin(2x) - \frac{1}{8}\sin(4x) - \frac{1}{2}u + \frac{1}{6}u^3\right) + C$$

$$= \frac{1}{16}\left(x + \cancel{\sin(2x)} - \frac{1}{4}\sin(4x) - \cancel{\sin(2x)} + \frac{1}{3}\sin^3(2x)\right) + C$$

$$= \frac{1}{16}\left(x - \frac{1}{4}\sin(4x) + \frac{1}{3}\sin^3(2x)\right) + C.$$

Eliminating Square Roots

Eg: Evaluate

$$\int_0^{\pi/4} \sqrt{1+\cos(4x)}\, dx$$

$$\sqrt{1+\cos(4x)} = \sqrt{2\left(\frac{1+\cos(2(2x))}{2}\right)} = \sqrt{2\cos^2(2x)} = \sqrt{2}\,\cos(2x)$$

$$\int_0^{\pi/4} \sqrt{1+\cos(4x)}\, dx = \sqrt{2}\int_0^{\pi/4} \cos(2x)\, dx$$

$$= \frac{\sqrt{2}}{2}\left[\sin(2x)\right]_0^{\pi/4}$$

$$= \frac{\sqrt{2}}{2}\left(\sin(\pi/2) - \sin(0)\right)$$

$$= \frac{\sqrt{2}}{2}(1-0)$$

$$= \frac{\sqrt{2}}{2}.$$

Powers of Secant & Tangent

Recall: $\tan^2(x) = \sec^2(x) - 1$

E.g: Evaluate $\int \tan^4(x)\,dx$. $\tan^4(x) = \tan^2(x)\tan^2(x)dx$.

$$\int \tan^4(x)\,dx = \int (\sec^2(x)-1)\tan^2(x)\,dx$$

$$= \int \sec^2(x)\tan^2(x)\,dx - \int \tan^2(x)\,dx$$

$$= \int \sec^2(x)\tan^2(x)\,dx - \int (\sec^2(x)-1)\,dx$$

$$= \int \sec^2(x)\tan^2(x)\,dx + \int dx - \int \sec^2(x)\,dx$$

Recall $\frac{d}{dx}\tan(x) = \sec^2(x)$, $u = \tan(x)$, $du = \sec^2(x)dx$

$$\int \tan^4(x)\,dx = \int u^2\,du + x - \tan(x) + C$$

$$= \frac{1}{3}\tan^3(x) + x - \tan(x) + C.$$

E.g: Evaluate $\int \sec(x)\,dx$.

Observe:

$$\frac{d}{dx}\sec(x) = \frac{d}{dx}\cos^{-1}(x) \overset{\text{Chain Rule.}}{=} -\cos^{-2}(x)(-\sin(x))$$

$$= \frac{\sin(x)}{\cos^2(x)} = \frac{1}{\cos(x)}\,\frac{\sin(x)}{\cos(x)} = \sec(x)\tan(x).$$

$$\frac{d}{dx}\tan(x) = \sec^2(x).$$

Hence

$$\frac{d}{dx}(\sec(x)+\tan(x)) = \sec(x)\tan(x) + \sec^2(x)$$

$$= \sec(x)(\tan(x) + \sec(x))$$

$$\int \sec(x)\,dx = \int \frac{\overbrace{\sec(x)(\tan(x)+\sec(x))}^{du}}{\underbrace{\tan(x)+\sec(x)}_{u}}\,dx$$

Take $u = \tan(x) + \sec(x)$, $du = \sec(x)(\tan(x)+\sec(x))\,dx$

$$\int \sec(x)\,dx = \int \frac{du}{u} = \ln|u| + C = \ln|\tan(x)+\sec(x)| + C.$$

E.g: $\int \sec^3(x)\,dx$

By parts:

$u = \sec(x)$ $\qquad$ $v = \tan(x)$

$du = \sec(x)\tan(x)\,dx$ $\qquad$ $dv = \sec^2(x)\,dx$

$$\underline{\int \sec^3(x)\,dx} = \sec(x)\tan(x) - \int \tan(x)(\sec(x)\tan(x))\,dx$$

$$= \sec(x)\tan(x) - \int \tan^2(x)\sec(x)\,dx$$

$$= \sec(x)\tan(x) - \int (\sec^2(x) - 1)\sec(x)\,dx$$

$$= \sec(x)\tan(x) - \underline{\int \sec^3(x)\,dx} - \int \sec(x)\,dx$$

$$2\int \sec^3(x)\,dx = \sec(x)\tan(x) - \ln|\sec(x) + \tan(x)| + C$$

$$\Rightarrow \int \sec^3(x)\,dx = \frac{\sec(x)\tan(x)}{2} - \frac{\ln|\sec(x) + \tan(x)|}{2} + C$$

Products of Sines and Cosines

$$\sin(mx)\sin(nx) = \tfrac{1}{2}\left[\cos((m-n)x) - \cos((m+n)x)\right]$$
$$\sin(mx)\cos(nx) = \tfrac{1}{2}\left[\sin((m-n)x) + \sin((m+n)x)\right]$$
$$\cos(mx)\cos(nx) = \tfrac{1}{2}\left[\cos((m-n)x) + \cos((m+n)x)\right]$$

Eg.: Evaluate $\int \sin(3x)\cos(5x)\,dx$

$$\int \sin(3x)\cos(5x)\,dx = \int \tfrac{1}{2}\left[\sin((3-5)x) + \sin((3+5)x)\right]dx$$
$$= \int \tfrac{1}{2}\left[\sin(-2x) + \sin(8x)\right]dx$$
$$= \tfrac{1}{2}\int \sin(-2x)\,dx + \tfrac{1}{2}\int \sin(8x)\,dx$$

sine is odd

$$= \tfrac{-1}{2}\int \sin(2x)\,dx + \tfrac{1}{2}\int \sin(8x)\,dx$$
$$= \tfrac{-1}{2}\left(\tfrac{-1}{2}\right)\cos(2x) + \tfrac{1}{2}\left(\tfrac{-1}{8}\right)\cos(8x) + C$$
$$= \tfrac{1}{4}\cos(2x) - \tfrac{1}{16}\cos(8x) + C.$$