# 8.2 : Integration by Parts

We want to integrate things that look like

$$\int f(x)g(x)\,dx$$

$$\frac{d}{dx}\left(f(x)g(x)\right) = f'(x)g(x) + f(x)g'(x)$$

$$\int \frac{d}{dx} f(x)g(x)\,dx = \int [f'(x)g(x) + f(x)g'(x)]\,dx$$

$$\parallel$$

$$f(x)g(x) = \int f'(x)g(x)\,dx + \int f(x)g'(x)\,dx$$

Let $u = f(x)$, $v = g(x)$, $\frac{du}{dx} = f'(x) \Rightarrow du = f'(x)dx$

$\frac{dv}{dx} = g'(x) \Rightarrow dv = g'(x)dx$

So

$$uv = \int v\,du + \int u\,dv$$

by subtracting $\int v\,du$ from both sides,

$$\int u\,dv = uv - \int v\,du.$$ Integration by parts formula.

E.g.: $\int x\cos(x)dx$ $\quad\int u\,dv = uv - \int v\,du$

$u = x$ $\qquad v = \cancel{\cos(x)}\ \sin(x)$

$du = dx$ $\qquad dv = \cos(x)dx$

$$\int \underbrace{x}_{u}\underbrace{\cos(x)dx}_{dv} = x\sin(x) - \int \underbrace{\sin(x)}_{v}\underbrace{dx}_{du}$$

$$= x\sin(x) - (-\cos(x)) + C$$

$$= x\sin(x) + \cos(x) + C.$$

E.g.: $\int x^2 e^x dx$

$u = x^2$ $\qquad v = e^x$

$du = 2x\,dx$ $\qquad dv = e^x dx$

$$\int x^2 e^x dx = x^2 e^x - \int e^x (2x\,dx)$$

$$= x^2 e^x - 2\int x e^x dx$$

Use integration by parts on $\int x e^x dx$

$u = x$ $\qquad v = e^x$

$du = dx$ $\qquad dv = e^x dx$

$$\int x e^x dx = xe^x - \int e^x dx = xe^x - e^x$$

$$\int x^2 e^x dx = x^2 e^x - 2(xe^x - e^x) + C = x^2e^x - 2xe^x + 2e^x + C$$

$$= e^x(x^2 - 2x + 2) + C.$$

Eg.: (Tricky) Integrate $e^x \sin(x)$.



$$u = e^x \qquad v = -\cos(x)$$
$$du = e^x dx \qquad du = \sin(x)\,dx$$

$$\int e^x \sin(x)\,dx = -e^x \cos(x) - \int (-\cos(x))\,e^x dx$$
$$= -e^x \cos(x) + \underline{\int e^x \cos(x)\,dx}$$

$$u = e^x \qquad v = \sin(x)$$
$$du = e^x dx \qquad du = \cos(x)\,dx$$

$$\int e^x \cos(x)\,dx = \underline{e^x \sin(x) - \int \sin(x)\,e^x dx}$$

$$\Rightarrow \int e^x \sin(x)\,dx = -e^x \cos(x) + e^x \sin(x) - \int e^x \sin(x)\,dx$$

Add $\int e^x \sin(x)\,dx$ to both sides, get

$$2\int e^x \sin(x)\,dx = -e^x \cos(x) + e^x \sin(x)$$

Divide both sides by 2 to get

$$\int e^x \sin(x)\,dx = \frac{-e^x \cos(x) + e^x \sin(x)}{2} + C.$$

Eg: Integrate $\ln(x)$.

$$\int \ln(x)\,dx$$

$u = \ln(x) \qquad v = x$

$du = \frac{1}{x}dx \qquad dv = dx$

$$\int \ln(x)\,dx = x\ln(x) - \int x\left(\tfrac{1}{x}\right)dx$$
$$= x\ln(x) - \int dx$$
$$= x\ln(x) - x + C.$$

Eg: (Reduction Formula) Evaluate

$$\int \cos^n(x)\,dx$$

$u = \cos^{n-1}(x) \qquad v = \sin(x)\cancel{dx}$

$\cancel{du = (n-1)\cos^{n-2}(x)dx} \qquad dv = \cos(x)\,dx$

$-(n-1)\cos^{n-2}(x)\sin(x)\,dx$

$$\int \cos^n(x)\,dx = \cos^{n-1}(x)\sin(x) + \int \sin^2(x)(n-1)\cos^{n-2}(x)\,dx$$
$$= \cos^{n-1}(x)\sin(x) + (n-1)\int \sin^2(x)\cos^{n-2}(x)\,dx$$
$$= \cos^{n-1}(x)\sin(x) + (n-1)\int (1-\cos^2(x))\cos^{n-2}(x)\,dx$$
$$= \cos^{n-1}(x)\sin(x) + (n-1)\int \left[\cos^{n-2}(x) - \cos^n(x)\right]dx$$

$\int \cos^n(x)dx = \cos^{n-1}(x)\sin(x) + (n-1)\int [\cos^{n-2}(x) - \cos^n(x)]dx$ 5

$= \cos^{n-1}(x)\sin(x) + (n-1)\int \cos^{n-2}(x)dx - (n-1)\int \cos^n(x)dx$

Add $(n-1)\int \cos^n(x)dx$ to both sides to get

$$[(n-1)+1]\int \cos^n(x)dx = \cos^{n-1}(x)\sin(x) + (n-1)\int \cos^{n-2}(x)dx$$

$$\Rightarrow n\int \cos^n(x)dx = \cos^{n-1}(x)\sin(x) + (n-1)\int \cos^{n-2}(x)dx$$

$$\Rightarrow \int \cos^n(x)dx = \frac{\cos^{n-1}(x)\sin(x)}{n} + \frac{n-1}{n}\int \cos^{n-2}(x)dx.$$

$\underline{n=1}$

$$\int \cos(x)dx = \sin(x) + C$$

$\underline{n=2}$

$$\int \cos^2(x)dx = \frac{\cos^{2-1}(x)\sin(x)}{2} + \frac{2-1}{2}\int \cos^{2-2}(x)dx$$

$$= \frac{\cos(x)\sin(x)}{2} + \frac{1}{2}\int dx$$

$$= \frac{\cos(x)\sin(x)}{2} + \frac{1}{2}x + C.$$

$\underline{n=4}$

$$\int \cos^4(x)dx = \frac{\cos^{4-1}(x)\sin(x)}{4} + \frac{4-1}{4}\int \cos^{4-2}(x)dx$$

$$= \frac{\cos^3(x)\sin(x)}{4} + \frac{3}{4}\int \cos^2(x)dx$$

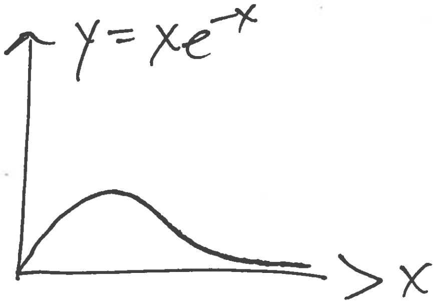
$$= \frac{\cos^3(x)\sin(x)}{4} + \frac{3}{4}\left(\frac{\cos(x)\sin(x)}{2} + \frac{1}{2}x\right) + C.$$

$$\int \cos^4(x)\,dx = \frac{\cos^3(x)\sin(x)}{4} + \frac{3\cos(x)\sin(x)}{8} + \frac{3}{8}x + C.$$ 6

## Definite Integrals

$$\int_a^b u\,dv = [uv]_a^b - \int_a^b v\,du.$$

Eg: Find the area of the region bounded by the curve $y = x\cdot e^{-x}$ and the x-axis from $x=0$, to $x=4$.




The area we want is given by the integral

$$\int_0^4 xe^{-x}\,dx = x(-e^{-x})\Big|_0^4 - \int_0^4 -e^{-x}\,dx$$

$u=x \quad v=-e^{-x}$

$du=dx \quad dv=e^{-x}dx$

$$= -xe^{-x}\Big|_0^4 + \int_0^4 e^{-x}\,dx$$

$$= -xe^{-x}\Big|_0^4 - e^{-x}\Big|_0^4$$

$$= -(4e^{-4}-0) - (e^{-4}-1)$$

$$= -4e^{-4} - e^{-4} + 1.$$

E.g.: $\int_0^{\pi/2} \theta^2 \sin(2\theta)\,d\theta$

$u = \theta^2 \qquad v = -\frac{1}{2}\cos(2\theta)$

$du = 2\theta\,d\theta \qquad dv = \sin(2\theta)\,d\theta$

$$\int_0^{\pi/2} \theta^2 \sin(2\theta)\,d\theta = -\frac{1}{2}\theta^2\cos(2\theta)\Big|_0^{\pi/2} - \int_0^{\pi/2}\left(-\frac{1}{2}\right)\cos(2\theta)(2\theta)\,d\theta$$

$$= -\frac{1}{2}\left[\frac{\pi^2}{4}\cos(\pi) - 0\right] + \int_0^{\pi/2}\theta\cos(2\theta)\,d\theta$$

$$= +\frac{1}{2}\frac{\pi^2}{4} + \underline{\int_0^{\pi/2}\theta\cos(2\theta)\,d\theta}$$

$u = \theta \qquad v = \frac{1}{2}\sin(2\theta)$

$du = d\theta \qquad dv = \cos(2\theta)\,d\theta$

$$\int_0^{\pi/2}\theta\cos(2\theta)\,d\theta = \frac{1}{2}\theta\sin(2\theta)\Big|_0^{\pi/2} - \frac{1}{2}\int_0^{\pi/2}\sin(2\theta)\,d\theta$$

$$= \frac{1}{2}\left[\pi/2\,\sin(\pi) - \frac{1}{2}0\,\sin(0)\right] - \frac{1}{2}\left(-\frac{1}{2}\right)\cos(2\theta)\Big|_0^{\pi/2}$$

$$= \frac{1}{4}\left[\cos(\pi) - \cos(0)\right]$$

$$= \frac{1}{4}(-1-1) = \frac{1}{4}(-2) = \underline{-\frac{1}{2}}.$$

$$\int_0^{\pi/2}\theta^2\sin(2\theta)\,d\theta = \frac{\pi^2}{8} - \frac{1}{2} = \frac{\pi^2}{8} - \frac{4}{8} = \frac{\pi^2-4}{8}.$$