The Fundamental Theorem of Calc

1/16/18 ①

Def^{n} ① A function f is continuous at a number a if

$$\lim_{x \to a} f(x) = f(a)$$

② A function f is continuous on an interval, I, if for every $a \in I$ (e.g. $I = [c, d]$, $c \le a \le d$), then

$$\lim_{x \to c^+} f(x) = f(c)$$

and

$$\lim_{x \to d^-} f(x) = f(d)$$

$$\lim_{x \to a} f(x) = f(a).$$

Thm: If f is a differentiable function, then f is continuous.

<u>Thm</u> (Fundamental Theorem of Calculus):

If f is a continuous function on $[a,b]$,

then the function

$$g(x) = \int_a^x f(t)\,dt, \quad a \le x \le b$$

is continuous on $[a,b]$, differentiable on (a,b),

and

$$g'(x) = f(x).$$

Pf: Let x, $x+h$ be such that

$$a \le x \le b, \quad a \le x+h \le b$$

we observe that

$$g(x+h) - g(x) = \int_a^{x} f(t)\,dt + \int_x^{x+h} f(t)\,dt - \int_a^x f(t)\,dt$$

$$= \int_x^{x+h} f(t)\,dt$$

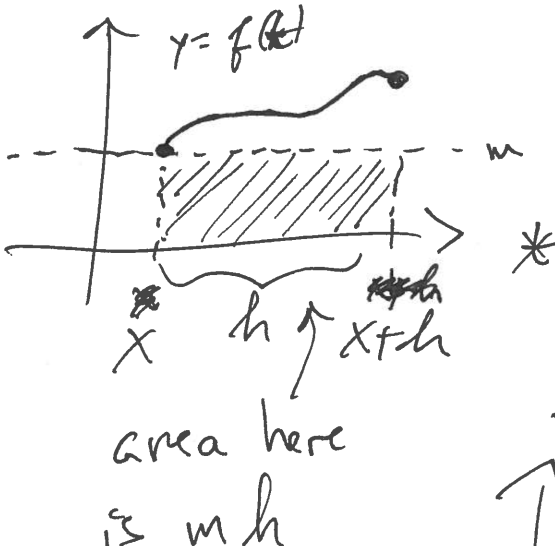
So, as long as $h \ne 0$

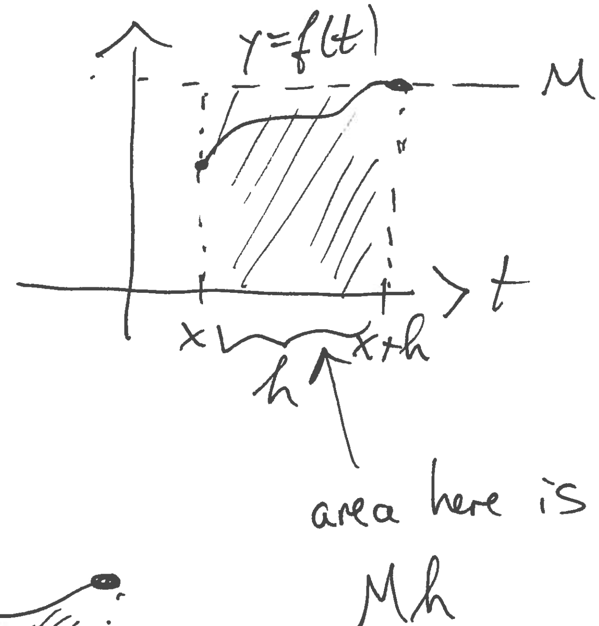
$$\frac{g(x+h) - g(x)}{h} = \frac{1}{h}\int_x^{x+h} f(t)\,dt$$

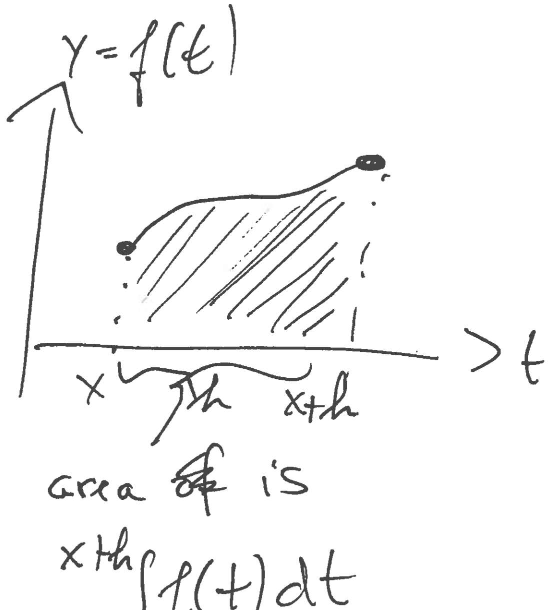
by definition

$$\lim_{h \to 0} \frac{g(x+h) - g(x)}{h} = g'(x)$$

<u>Recall:</u> (Extreme Value Theorem): If f is continuous on a closed interval, $[x, x+h]$, then for some number $x \le u \le x+h$, f attains its minimum, $m = f(u)$, and for some number, v, $x \le v \le x+h$, f attains its maximum, $M = f(v)$.







So when $h > 0$, we can bound

$$mh \le \int_x^{x+h} f(t)\,dt \le Mh$$

We want to know what happens as $h \to 0$. As we do this, $x+h \to x$, but also $u \to x$, and $v \to x$. By continuity

$$\lim_{h \to 0} f(u) = \lim_{u \to x} f(u) = f(x)$$

and

$$\lim_{h \to 0} f(v) = \lim_{v \to x} f(v) = f(x).$$

<u>Recall</u>: (Squeeze Thm): If $f(x) \le g(x) \le h(x)$ near a and

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$$

then $\lim_{x \to a} g(x) = L$.

Apply the squeeze theorem to

$$m = f(u) \le \frac{g(x+h) - g(x)}{h} = \frac{1}{h}\int_x^{x+h} f(t)\,dt \le M$$

to get

$$\lim_{h \to 0} \frac{g(x+h) - g(x)}{h} = f(x).$$

Remarks: The function g is not necessarily differentiable at the endpoints, a and b.

$$g(x) = \int_a^x f(t)\,dt \qquad a \leq x \leq b$$

but it is left/right continuous.

E.g: Find

$$\frac{d}{dx}\int_1^{x^4} \sec(t)\,dt$$

$$g(x) = \int_1^{x^4} \sec(t)\,dt, \quad u = x^4 \qquad \frac{du}{dx} = 4x^3$$

$$g(u) = \int_1^{u} \sec(t)\,dt$$

$$\frac{d}{dx} g^*(u) \overset{\text{Chain Rule}}{=} g'(u)\frac{du}{dx} \overset{\text{FTC}}{=} \sec(u)\frac{du}{dx}$$

$$= \sec(x^4)(4x^3)$$

$$= 4x^3\sec(x^4).$$

Thm (FTC): If f is continuous on $[a,b]$ and F is any anti-derivative of f, then

$$\int_a^b f(x)dx = F(b) - F(a).$$

Substitution

Want to integrate something of the form

$$\int f'(g(x))\, g'(x)\, dx = f(g(x))$$

E.g.: $\int x^3 \cos(x^4+2)\, dx$

$u = x^4 + 2$

$\frac{du}{dx} = 4x^3$

$\Rightarrow du = 4x^3 dx$

$\Rightarrow \frac{1}{4} du = x^3 dx$

$$= \int \cos(u) \tfrac{1}{4} du = \frac{1}{4}\int \cos(u) du = \frac{1}{4}\sin(u) + c = \frac{1}{4}\sin(x^4+2) + c.$$

Definite Integrals

If g' ~~can~~ is continuous on $[a,b]$ and f is continuous on the range of $u = g(x)$, then

$$\int_a^b f'(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$$

E.g: $\int_0^4 \sqrt{2x+1}\,dx$

$u = 2x+1$

$\frac{du}{dx} = 2$

$\Rightarrow du = 2dx$

$\Rightarrow \frac{1}{2}du = dx$

$$= \int_1^9 \sqrt{u}\left(\tfrac{1}{2}du\right)$$

$$= \frac{1}{2}\int_1^9 u^{1/2}\,du = \frac{1}{2}\,u^{3/2}\Big|_1^9\left(\frac{2}{3}\right) = \frac{1}{3}\left(9^{3/2} - 1^{3/2}\right)$$

$$= \frac{1}{3}\left(3^3 - 1\right)$$

$$= \frac{1}{3}(27-1)$$

$$= \frac{1}{3}26 = \frac{26}{3}.$$

E.g: $\int_1^e \frac{\ln(x)}{x}\,dx = \int_{u(1)}^{u(e)} u\,du$

$u = \ln(x)$

$\frac{du}{dx} = \frac{1}{x}$

$\Rightarrow du = \frac{dx}{x}$

$$= \int_0^1 u\,du$$

$$= \frac{1}{2}u^2\Big|_0^1 = \frac{1}{2}\left(1^2 - 0^2\right) = \frac{1}{2}.$$

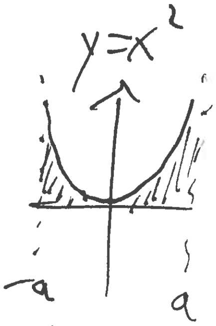
Recall: A function, f, is odd if $f(-x) = -f(x)$

" even if $f(-x) = f(x)$.

If f is cont. on $[a, a]$

① If f is even, then

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$



② If f is odd, then

$$\int_{-a}^{a} f(x)\,dx = 0$$