MATH 142: EXAM 02

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Answer the questions in the spaces provided on the question sheets and turn them in at the end of the class period. Unless otherwise stated, all supporting work is required. It is advised, although not required, that you check your answers. You may **not** use any calculators.

Name: Answer Key

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1. Problems

1. *(a) Compute* $\int_{-\infty}^{\infty} xe^{-x^2}dx$

$$\int_0^\infty xe^{-x^2}dx = \lim_{t\to\infty}\int_0^t xe^{-x^2}dx \qquad u=-x^2, \quad du=-2xdx, \quad -\tfrac{1}{2}du = xdx$$

$$= \lim_{t\to\infty} -\frac{1}{2}\int_0^{-t^2} e^u du$$

$$= \lim_{t\to\infty} -\frac{1}{2}\left(e^{-t^2}-e^0\right)$$

$$= \lim_{t\to\infty} -\frac{1}{2}\left(e^{-t^2}-1\right)$$

$$= -\frac{1}{2}(0-1)$$

$$= \frac{1}{2}.$$

$$\int_{-\infty}^0 xe^{-x^2}dx = \lim_{t\to-\infty}\int_t^0 xe^{-x^2}dx = \lim_{t\to-\infty} -\frac{1}{2}\int_{-t^2}^0 e^u du = \lim_{t\to-\infty} -\frac{1}{2}\left(e^0-e^{-t^2}\right) = -\frac{1}{2}(1-0) = -\frac{1}{2}.$$

$$\int_{-\infty}^\infty xe^{-x^2}dx = \int_0^\infty xe^{-x^2}dx + \int_{-\infty}^0 xe^{-x^2}dx = \frac{1}{2}-\frac{1}{2} = 0.$$

(b) Does the series $\sum_{n=0}^{\infty} ne^{-n^2}$ *converge or diverge? Justify your answer.*

$\sum_{n=0}^{\infty} ne^{-n^2}$ converges by the Integral Test because

$\int_0^\infty xe^{-x^2}dx = 1/2$ converges.

2. *Express the decimal* $0.\overline{9} = 0.999999\ldots$ *as a rational number. [Hint: Geometric Series.]*

$$0.\overline{9} = \sum_{n=1}^{\infty} 9/10^n = \sum_{n=0}^{\infty} \frac{9}{10}\frac{1}{10^n} = \frac{\frac{9}{10}}{1-\left(\frac{1}{10}\right)} = \frac{\frac{9}{10}}{9/10} = \boxed{1}.$$

3. *Find the radius of convergence and the interval of convergence for the power series* $\sum_{n=1}^{\infty} \frac{x^n}{\sqrt{n}}$.

$$\lim_{n\to\infty} \left| \frac{x^{n+1}}{\sqrt{n+1}} \frac{\sqrt{n}}{x^n} \right| = \lim_{n\to\infty} |x| \sqrt{\frac{n}{n+1}}$$

$$= |x| \sqrt{\lim_{n\to\infty} \frac{n}{n+1}}$$

$$= |x| \sqrt{1}$$

$$= |x| < 1$$

so the radius of convergence is 1 by the Ratio Test.

$\underline{x=-1}$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$$

(i) $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$

(ii) $\sqrt{n} \leq \sqrt{n+1} \Rightarrow \frac{1}{\sqrt{n+1}} \leq \frac{1}{\sqrt{n}}$

so $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ converges by the A.S.T.

$\underline{x=1}$

$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is a divergent p-series $(p=\frac{1}{2}<1)$.

The interval of convergence is then

$[-1, 1)$.

Test the following series for convergence. You may use any of the tests we covered in class, however you **must indicate which test you use**.

4. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2-1}{3n^4+1}$.

Use limit comparison with

$$\sum_{n=1}^{\infty} \frac{n^2}{n^4} = \sum_{n=1}^{\infty} \frac{1}{n^2},$$

which is a convergent p-series

$$\lim_{n\to\infty} \frac{n^2-1}{3n^4+1}\left(\frac{n^2}{1}\right) = \lim_{n\to\infty} \frac{n^4-n^2}{3n^4+1} = \frac{1}{3},$$

so both series converge.

5. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n^2}{n^3+4}$

(i) $\displaystyle\lim_{n\to\infty}\frac{n^2}{n^3+4}=0$

(ii) Let $f(x)=x^2/x^3+4$. Then

$$f'(x)=\frac{2x(x^3+4)-x^2(3x^2)}{(x^3+4)^2}$$

$$=\frac{2x^4+8x-3x^4}{(x^3+4)^2}$$

$$=\frac{8x-x^4}{(x^3+4)^2}$$

$$=\frac{x(8-x^3)}{(x^3+4)^2}=0 \iff x=0 \text{ or } x=2.$$

When $2<x$, $0<x$, $(x^3+4)^2>0$, and

$$8<x^3 \Rightarrow 8-x^3<0,$$

so $f'(x)<0$.

Therefore

$$a_{n+1}=\frac{(n+1)^2}{(n+1)^3+4}<\frac{n^2}{n^3+4}=a_n$$

whenever $n>2$, so by the A.S.T.

$$\sum_{n=3}^{\infty}\frac{(-1)^{n+1}n^2}{n^3+4}<\infty$$

and so

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n^2}{n^3+4}=\frac{1}{5}-\frac{1}{3}+\sum_{n=3}^{\infty}(-1)^{n+1}\frac{n^2}{n^3+4}<\infty.$$

Therefore the series converges by the A.S.T.

6. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{2^n}{n^2}$

This series diverges by the Divergence Test:

$$\lim_{n\to\infty}\frac{2^n}{n^2}\overset{\text{L'Hôpital}}{=}\lim_{n\to\infty}\frac{\ln(2)\,2^n}{2n}$$

$$\overset{\text{L'Hôpital}}{=}\lim_{n\to\infty}\frac{(\ln(2))^2\,2^n}{2}$$

$$=\infty.$$

The even terms of $(-1)^{n-1}2^n/n^2$ approach $-\infty$, the odd terms approach $+\infty$, so

$$\lim_{n\to\infty}(-1)^{n-1}2^n/n^2$$

does not exist.

7. $\sum_{n=1}^{\infty} \left(\frac{3}{n}\right)^n$

By the Root Test

$$\lim_{n\to\infty} \sqrt[n]{\left(\frac{3}{n}\right)^n} = \lim_{n\to\infty} \frac{3}{n} = 0 < 1$$

this series converges.