MATH 122

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Calculus for Business Administration and Social Sciences

Outline

1. 5.3: Area Between Two Curves

Motivation

Say we wanted to find the area between the two curves

$$f(x) = x^2 \text{ and } g(x) = x$$

on the interval $[0, 1]$.

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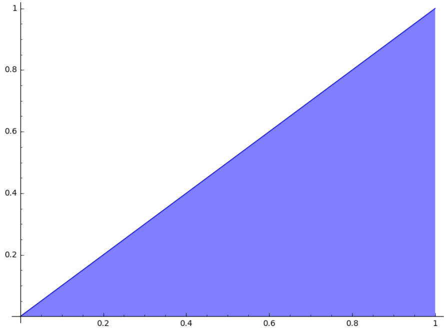
$$f(x) = x^2 \text{ and } g(x) = x$$

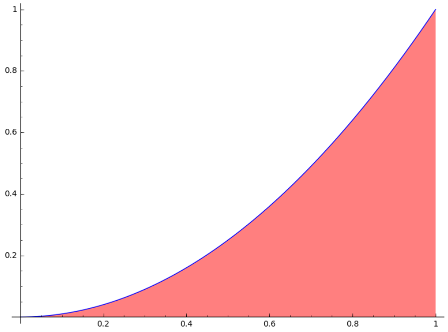
on the interval $[0, 1]$. Geometrically, this is obvious: compute the bigger area, then subtract the smaller area.

Motivation (Cont.)

The area under the line:

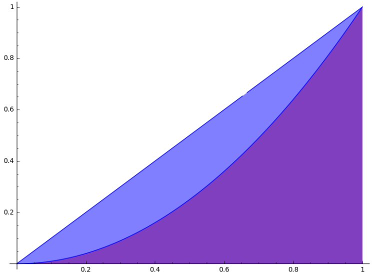


The area under the parabola:



Motivation (Cont.)

On one plot:



The blue area is what we want to compute. The purple area is the intersection of the red solid and the blue solid; this is the area we want to remove from the area under the line.

Motivation (Cont.)

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$$\int_0^1 x\,\mathrm{d}x - \int_0^1 x^2\,\mathrm{d}x \quad = \quad \frac{1}{2}x^2\Big|_0^1 - \frac{1}{3}x^3\Big|_0^1$$

Motivation (Cont.)

So, all we need to do is compute

$$\begin{aligned}\int_0^1 x\,dx - \int_0^1 x^2\,dx &= \frac{1}{2}x^2\Big|_0^1 - \frac{1}{3}x^3\Big|_0^1\\ &= \frac{1}{2}(1-0) - \frac{1}{3}(1-0)\end{aligned}$$

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&= \frac{1}{6}.
\end{aligned}$$

Assume that f and g are continuous functions on $[a, b]$ and that $g(x) \leq f(x)$ for all $a \leq x \leq b$. The area between the two curves is

$$\int_a^b f(x) - g(x)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x - \int_a^b g(x)\,\mathrm{d}x.$$

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Find the area between $f(x) = 4x - x^2$ and $g(x) = \frac{1}{2}x^{\frac{3}{2}}$ for $0 \leq x$.

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implies either $x = 0$ or $4 - x = \frac{1}{2}\sqrt{x}$. The latter is equivalent to solving

$$2x + \sqrt{x} - 8 = 2\sqrt{x}^2 + \sqrt{x} - 8 = 0.$$

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Since $\sqrt{x}$ is positive, the only solution is

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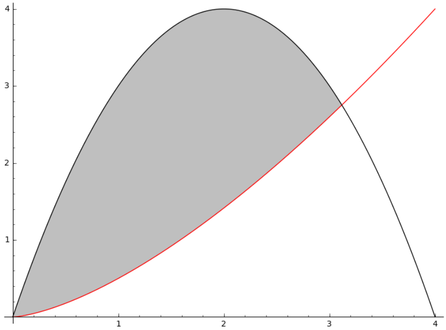
Hence

$$x = \left(\frac{-1+\sqrt{65}}{4}\right)^2 \approx 3.11$$

Example (Cont.)

The plot of the two functions is



The parabola, $f(x)$ is on top, and $g(x)$ is the bottom curve.

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$$\begin{aligned}
\int_0^b f(x) - g(x)\,dx &= \int_0^b 4x - x^2 - \frac{1}{2}x^{\frac{3}{2}}\,dx \\
&= 4\int_0^b x\,dx - \int_0^b x^2\,dx - \frac{1}{2}\int_0^b x^{\frac{3}{2}}\,dx
\end{aligned}$$

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&= 4\int_0^b x\,dx - \int_0^b x^2\,dx - \frac{1}{2}\int_0^b x^{\frac{3}{2}}\,dx \\
&= 2x^2\Big|_0^b - \frac{1}{3}x^3\Big|_0^b - \frac{1}{5}x^{\frac{5}{2}}\Big|_0^b
\end{aligned}$$

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&= 2x^2\Big|_0^b - \frac{1}{3}x^3\Big|_0^b - \frac{1}{5}x^{\frac{5}{2}}\Big|_0^b \\
&= 2\left(\frac{-1+\sqrt{65}}{4}\right)^4 - \frac{1}{3}\left(\frac{-1+\sqrt{65}}{4}\right)^6 - \frac{1}{5}\left(\frac{-1+\sqrt{65}}{4}\right)^5
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&= 2\left(\frac{-1+\sqrt{65}}{4}\right)^4 - \frac{1}{3}\left(\frac{-1+\sqrt{65}}{4}\right)^6 - \frac{1}{5}\left(\frac{-1+\sqrt{65}}{4}\right)^5 \\
&\approx 5.91.
\end{aligned}$$