Math 122

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Calculus for Business Administration and Social Sciences

Outline

1. 5.2: The Definite Integral

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The Definite Integral

- In the last section, we saw that for a continuous function f on an interval $[a, b]$, the error for Left-Hand Sums and Right-Hand Sums goes to zero as the number of points in the partition becomes large.

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- As the error term goes to zero, the Left-Hand Sum increases towards a fixed value and the Right-Hand Sum decreases towards that same value.
- The common value that these sums approach is called a *limit*, and we call this particular limit the Definite Integral.

Formal Definition of the Definite Integral

Definition 1

Assume that f is continuous on the interval $[a, b]$. The *definite integral of f from a to b* is

$$\int_a^b f(t)\,\mathrm{d}\,t = \lim_{n\to\infty}\sum_{i=0}^{n} f(t_i)\Delta t = \lim_{n\to\infty}\sum_{i=1}^{n} f(t_i)\Delta t,$$

where the set of t-values

$$a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b$$

is a partition of $[a, b]$ into n intervals of length

$$\Delta t = \frac{b-a}{n}.$$

Compute $\int_{x_0}^{x_1} b \, \mathrm{d}\, x$ for $0 < b$.

Geometric Examples

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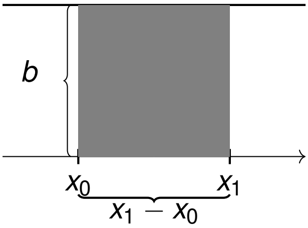

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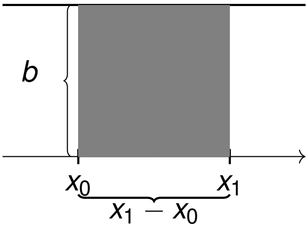

$$\int_{x_0}^{x_1} b \, \mathrm{d}\, x = b(x_1 - x_0).$$

Geometric Examples

Let $f(x) = mx + b$ for $0 < b$, $0 < m$. Compute $\int_{x_0}^{x_1} f(x)\,\mathrm{d}\,x$ for $\frac{-b}{m} < x_0$.

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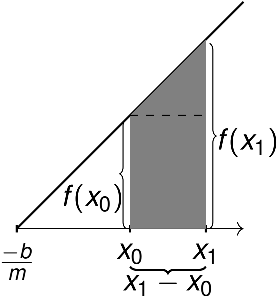

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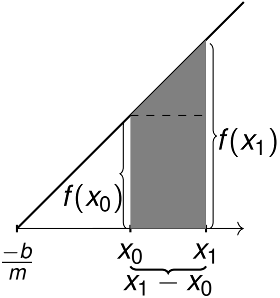

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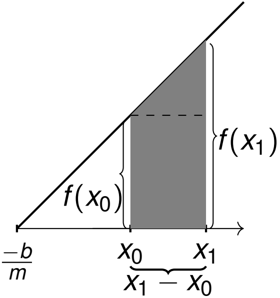

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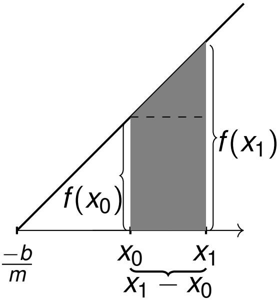

$$\int_{x_0}^{x_1} f(x)\,\mathrm{d}\,x = f(x_0)(x_1 - x_0) + \frac{1}{2}\left[f(x_1) - f(x_0)\right](x_1 - x_0).$$

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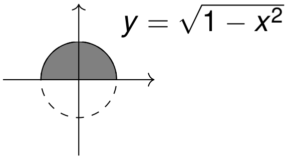

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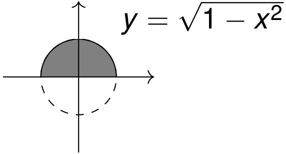
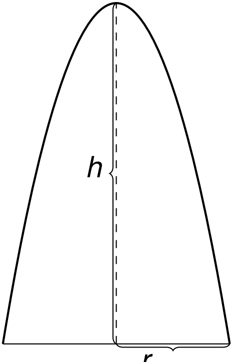


- Therefore

$$\int_{-1}^{1} \sqrt{1-x^2}\, \mathrm{d}\, x = \frac{1}{2}\pi(1)^2 = \frac{\pi}{2}.$$

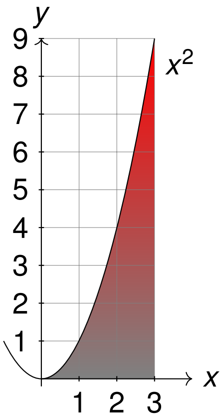
The area under the parabola



is $\frac{4}{3}rh$. Use this to compute $\int_0^3 x^2 \, dx$.

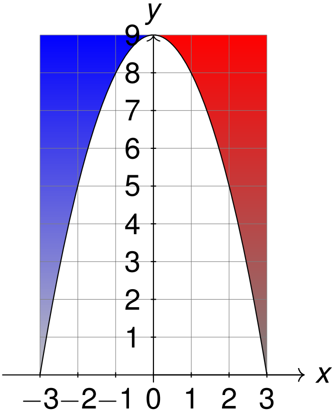
Geometric Examples

The integral is just the area under the parabola:



Geometric Examples

If we flip the picture upside down, we have the picture



And we note that the red and blue areas are, by symmetry, the same.

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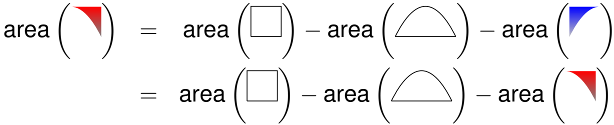

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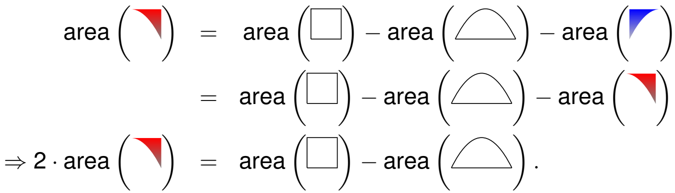

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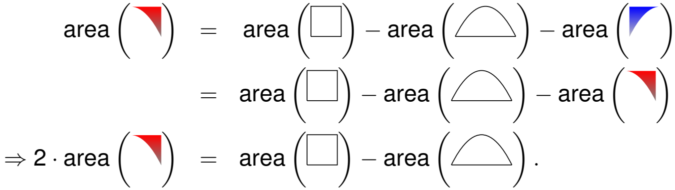

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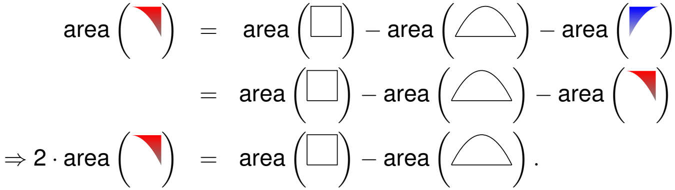

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$$\int_0^3 x^2 \, d\,x \quad =$$

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$\Rightarrow 2 \cdot$ area () $=$ area () $-$ area () .

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$$\int_0^3 x^2 \, \mathrm{d}\, x = \frac{1}{2}\left[6 \cdot 9 - \frac{4}{3} \cdot 9 \cdot 3\right]$$

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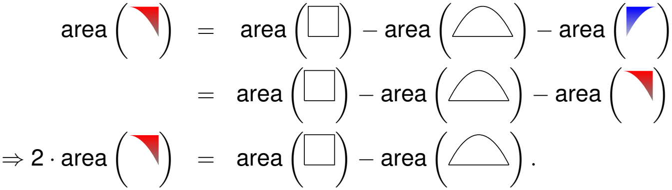

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$$= 9.$$

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THEOREM 1 (FUNDAMENTAL THEOREM OF CALCULUS)

If $F'(t)$ is a continuous function on $[a, b]$, then

$$\int_a^b F'(t)\,\mathrm{d}t = F(b) - F(a).$$

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Remark 1

Essentially, this says that the area between the derivative of F and the x-axis from a and b is just the total change in F on the interval $[a, b]$.

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