Math 122

Blake Farman [1]

[1]University of South Carolina, Columbia, SC USA

Calculus for Business Administration and Social Sciences

Outline

1. 5.1: Distance and Accumulated Change
   - Constant Functions
   - Linear Functions
   - Non-Linear Functions
   - Right Endpoint Estimates
   - Left Endpoint Estimates
   - Partitions
   - Left- and Right-Hand Sums
   - Applying Our Method

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This is easy:

$$60\ \frac{\text{miles}}{\text{hour}} \cdot 2\ \text{hours} = 120\ \text{miles}.$$

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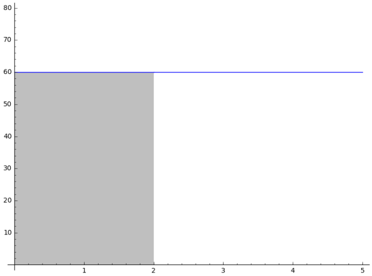

This says that under constant velocity, v, the position of the car, $s(t)$, relative to the starting point at time $0 \leq t$ is just

$$s(t) = v \cdot t.$$

According to Car and Driver, a 2006 Bugatti Veyron is capable of an acceleration of $11.59\,\mathrm{m}\,/\,\mathrm{s}^2$. Assume the car starts at rest and accelerates at this constant rate.

Linear Functions

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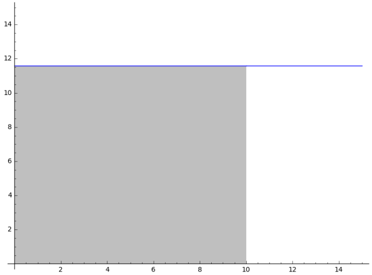

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$$s(t) = \frac{1}{2}v(t) \cdot t$$

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$$\begin{aligned} s(t) &= \frac{1}{2}v(t)\cdot t \\ &= \frac{1}{2}(11.59\cdot t)\cdot t \end{aligned}$$

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$$\begin{aligned} s(t) &= \frac{1}{2}v(t)\cdot t \\ &= \frac{1}{2}(11.59\cdot t)\cdot t \\ &= \frac{11.59}{2}t^2. \end{aligned}$$

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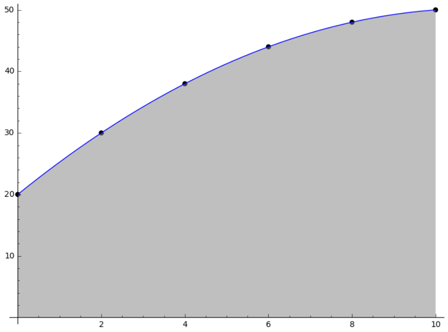

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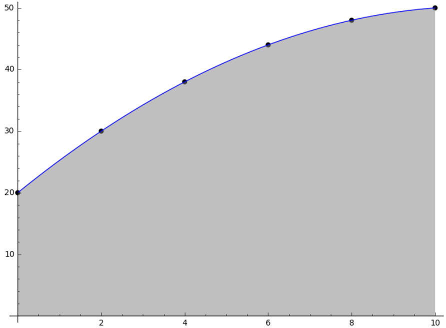

How do we compute the area of the shaded region?

We could assume constant velocity between the two points and estimate.

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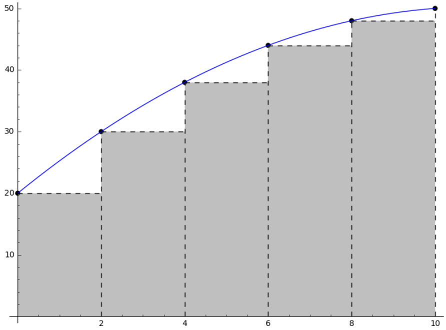

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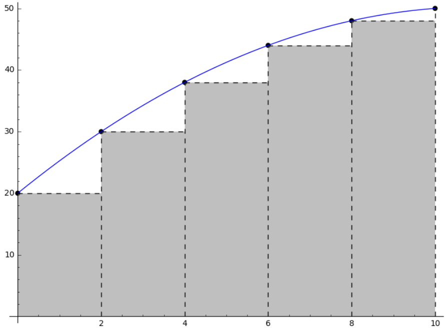
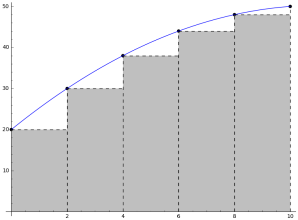


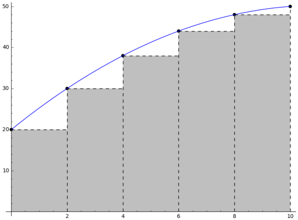
This is an underestimate of the area.

Non-Linear Functions



- Each rectangle has width 2.

Non-Linear Functions



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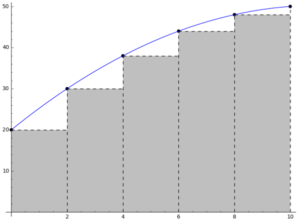

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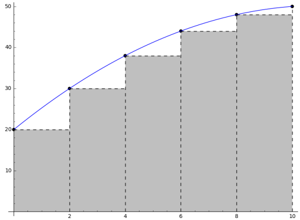

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$$2(20+30+38+44+48) \quad =$$

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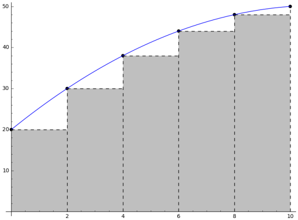

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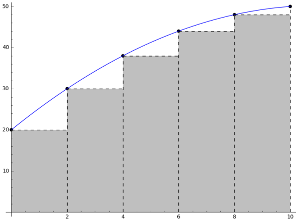
$$2(20 + 30 + 38 + 44 + 48) \quad = \quad 2(180)$$



- Each rectangle has width 2.
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- Our area estimate is:

$$\begin{aligned} 2(20+30+38+44+48) &= 2(180) \\ &= 360 \text{ feet.} \end{aligned}$$

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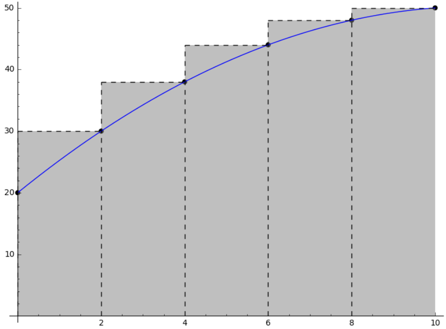

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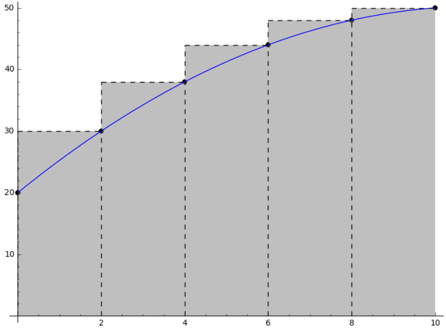

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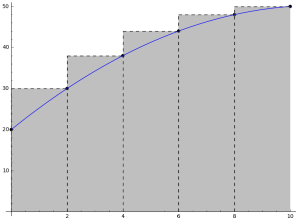

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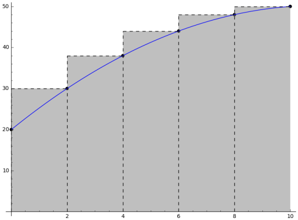

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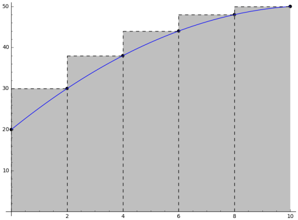

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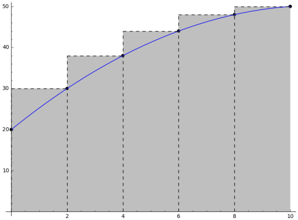

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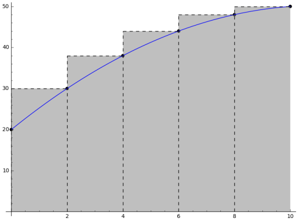

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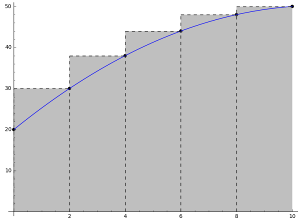
$$2(30 + 38 + 44 + 48 + 50) = 2(210)$$



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- Our area estimate is:

$$
\begin{aligned}
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&= 420 \text{ feet.}
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Can we do better? If so, how?

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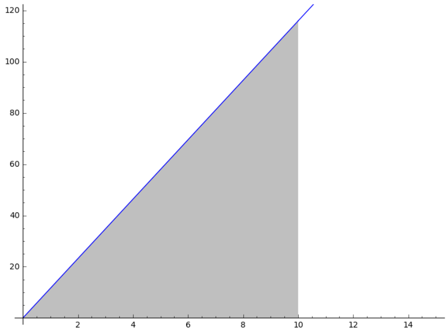

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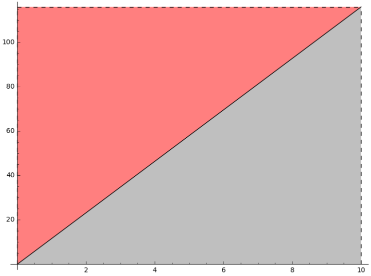

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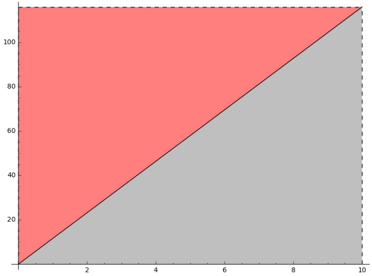

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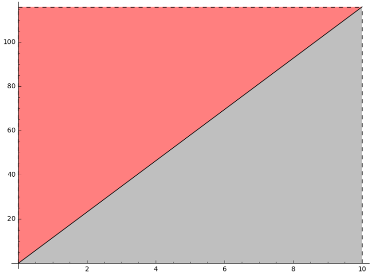

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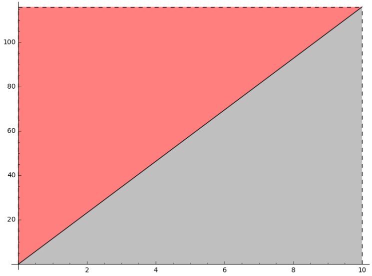

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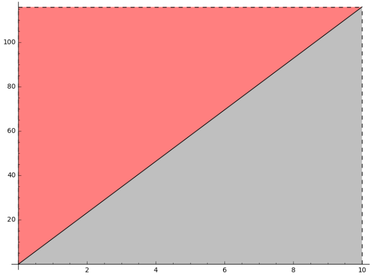

- Red is the error.
- Grey is the area.
- The estimate for the area is the sum of the red and grey areas.
- The error is equal to the actual area!

Three Equidistant Points

If we try three equidistant points, 0, $\frac{t}{2}$, and t, then we get:

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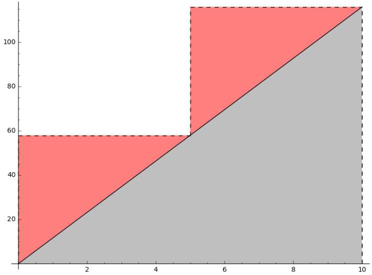

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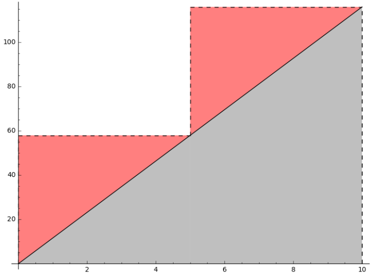

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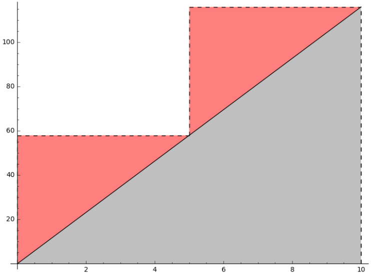

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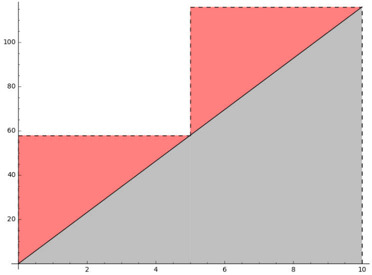

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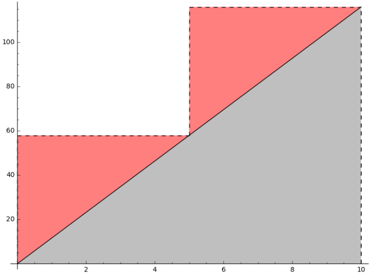

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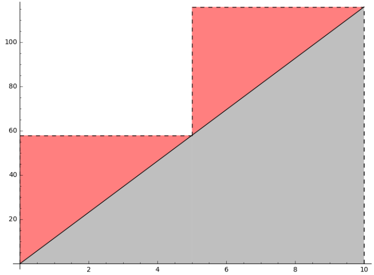

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- The error is the area of the two red triangles.
- Both have base length $\frac{t}{2}$; here $t = 10$.
- The height of the left triangle is $v\left(\frac{t}{2}\right)$.
- The height of the right triangle is $v(t) - v\left(\frac{t}{2}\right)$.

Three Equidistant Points (cont.)

So, the total error is:

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$$\frac{1}{2}\left[v(t) - v\left(\frac{t}{2}\right)\right]\frac{t}{2} +$$

Three Equidistant Points (cont.)

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$$\frac{1}{2}\left[v(t) - v\left(\frac{t}{2}\right)\right]\frac{t}{2} + \frac{1}{2}v\left(\frac{t}{2}\right)\cdot\frac{t}{2}$$

Three Equidistant Points (cont.)

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Three Equidistant Points (Cont.)

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$$\begin{aligned}\frac{1}{2}\left[v(t)-v\left(\frac{t}{2}\right)\right]\frac{t}{2}+\frac{1}{2}v\left(\frac{t}{2}\right)\cdot\frac{t}{2} &= \frac{1}{2}\left[v(t)-v\left(\frac{t}{2}\right)+v\left(\frac{t}{2}\right)\right]\frac{t}{2}\\ &= \frac{1}{2}\left(\frac{1}{2}v(t)\cdot t\right).\end{aligned}$$

Three Equidistant Points (cont.)

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$$\begin{aligned} \frac{1}{2}\left[v(t) - v\left(\frac{t}{2}\right)\right]\frac{t}{2} + \frac{1}{2}v\left(\frac{t}{2}\right)\cdot\frac{t}{2} &= \frac{1}{2}\left[v(t) - v\left(\frac{t}{2}\right) + v\left(\frac{t}{2}\right)\right]\frac{t}{2} \\ &= \frac{1}{2}\left(\frac{1}{2}v(t)\cdot t\right). \end{aligned}$$

By adding one more point, we've reduced the error by a factor of two!

Four Equidistant Points (Cont.)

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Four Equidistant Points (Cont.)

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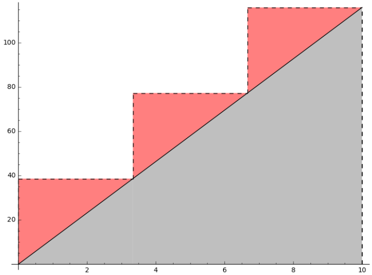

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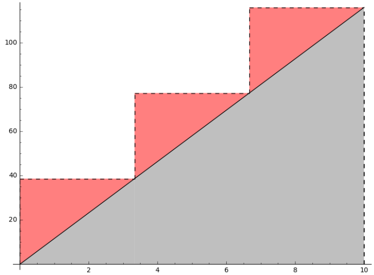

- Visibly, this is an even better estimate.

Four Equidistant Points (cont.)

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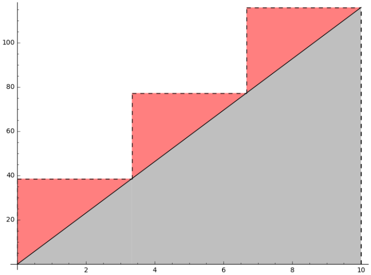

- Visibly, this is an even better estimate.
- All three red triangles have base length $\frac{t}{3}$.

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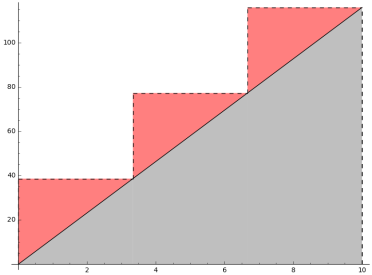

- Visibly, this is an even better estimate.
- All three red triangles have base length $\frac{t}{3}$.
- The height of the left triangle is $v\left(\frac{t}{3}\right)$.

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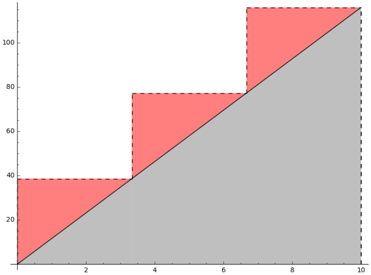

- Visibly, this is an even better estimate.
- All three red triangles have base length $\frac{t}{3}$.
- The height of the left triangle is $v\left(\frac{t}{3}\right)$.
- The height of the middle triangle is $v\left(\frac{2t}{3}\right) - v\left(\frac{t}{3}\right)$.

Four Equidistant Points (Cont.)

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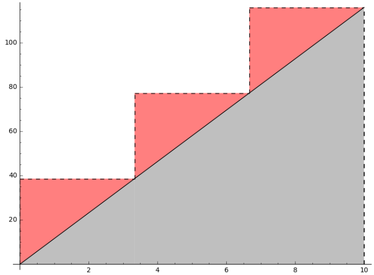

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- All three red triangles have base length $\frac{t}{3}$.
- The height of the left triangle is $v\left(\frac{t}{3}\right)$.
- The height of the middle triangle is $v\left(\frac{2t}{3}\right) - v\left(\frac{t}{3}\right)$.
- The height of the right triangle is $v(t) - v\left(\frac{2t}{3}\right)$.

Four Equidistant Points (cont.)

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$$\frac{1}{2}\left[v(t) - v\left(\frac{2t}{3}\right)\right]\frac{t}{3} +$$

Four Equidistant Points (cont.)

So, the total error is:

$$\frac{1}{2}\left[v(t) - v\left(\frac{2t}{3}\right)\right]\frac{t}{3} + \frac{1}{2}\left[v\left(\frac{2t}{3}\right) - v\left(\frac{t}{3}\right)\right]\frac{t}{3} +$$

Four Equidistant Points (Cont.)

So, the total error is:

$$\frac{1}{2}\left[v(t) - v\left(\frac{2t}{3}\right)\right]\frac{t}{3} + \frac{1}{2}\left[v\left(\frac{2t}{3}\right) - v\left(\frac{t}{3}\right)\right]\frac{t}{3} + \frac{1}{2}v\left(\frac{t}{3}\right)\frac{t}{3}$$

FOUR EQUIDISTANT POINTS (CONT.)

So, the total error is:

$$\frac{1}{2}\left[v(t) - v\left(\frac{2t}{3}\right)\right]\frac{t}{3} + \frac{1}{2}\left[v\left(\frac{2t}{3}\right) - v\left(\frac{t}{3}\right)\right]\frac{t}{3} + \frac{1}{2}v\left(\frac{t}{3}\right)\frac{t}{3} = \frac{1}{2}\left[v(t) - v\left(\frac{2t}{3}\right) + v\left(\frac{2t}{3}\right) - v\left(\frac{t}{3}\right) + v\left(\frac{t}{3}\right)\right]\frac{t}{3}$$

Four Equidistant Points (Cont.)

So, the total error is:

$$\begin{aligned}\frac{1}{2}\left[v(t)-v\left(\frac{2t}{3}\right)\right]\frac{t}{3}+\frac{1}{2}\left[v\left(\frac{2t}{3}\right)-v\left(\frac{t}{3}\right)\right]\frac{t}{3}+\frac{1}{2}v\left(\frac{t}{3}\right)\frac{t}{3} &= \frac{1}{2}\left[v(t)-v\left(\frac{2t}{3}\right)+v\left(\frac{2t}{3}\right)-v\left(\frac{t}{3}\right)+v\left(\frac{t}{3}\right)\right]\frac{t}{3}\\ &= \frac{1}{3}\left(\frac{1}{2}v(t)\cdot t\right).\end{aligned}$$

Four Equidistant Points (Cont.)

So, the total error is:

$$\frac{1}{2}\left[v(t) - v\left(\frac{2t}{3}\right)\right]\frac{t}{3} + \frac{1}{2}\left[v\left(\frac{2t}{3}\right) - v\left(\frac{t}{3}\right)\right]\frac{t}{3} + \frac{1}{2}v\left(\frac{t}{3}\right)\frac{t}{3} = \frac{1}{2}\left[v(t) - v\left(\frac{2t}{3}\right) + v\left(\frac{2t}{3}\right) - v\left(\frac{t}{3}\right) + v\left(\frac{t}{3}\right)\right]\frac{t}{3}$$
$$= \frac{1}{3}\left(\frac{1}{2}v(t)\cdot t\right).$$

By using four points, we've reduced the initial error by a factor of three!

$n+1$ Equidistant Points

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$$t_0 = 0,$$

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- base length $\frac{t}{n}$,
- height $v(t_k) - v(t_{k-1})$,
- area

$$\frac{1}{2}\left[v\left(t_k\right) - v\left(t_{k-1}\right)\right]\frac{t}{n}$$

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$$\frac{1}{2}\left[v\left(t_k\right) - v\left(t_{k-1}\right)\right]\frac{t}{n}$$

Remark 1

Note that $v(t_0) = v(0) = 0$.

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Adding up the areas of each of the triangles, we get the total error:

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Adding up the areas of each of the triangles, we get the total error:

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Adding up the areas of each of the triangles, we get the total error:

$$\frac{1}{2}\left[v(t)-v(t_{k-1})+v(t_{k-1})-v(t_{k-2})+\ldots+v(t_2)-v(t_1)+v(t_1)-v(t_0)\right]\frac{t}{n}$$

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Adding up the areas of each of the triangles, we get the total error:

$$\frac{1}{2}\left[v(t) - v(t_{k-1}) + v(t_{k-1}) - v(t_{k-2}) + \ldots + v(t_2) - v(t_1) + v(t_1) - v(t_0)\right]\frac{t}{n} = \frac{1}{2}v(t) \cdot \frac{t}{n}$$

$n+1$ Equidistant Points

Adding up the areas of each of the triangles, we get the total error:

$$\begin{aligned}\frac{1}{2}\left[v(t)-v(t_{k-1})+v(t_{k-1})-v(t_{k-2})+\ldots+v(t_2)-v(t_1)+v(t_1)-v(t_0)\right]\frac{t}{n} &= \frac{1}{2}v(t)\cdot\frac{t}{n}\\ &= \frac{1}{n}\left(\frac{1}{2}v(t)\cdot t\right).\end{aligned}$$

$n+1$ Equidistant Points

Adding up the areas of each of the triangles, we get the total error:

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Therefore, if we use $n+1$ equidistant points, we have overestimated the area under $v(t)$ by

$$\frac{1}{n}\left(\frac{1}{2}v(t)\cdot t\right).$$

Left Estimate

The situation for a left endpoint estimate is symmetric:

Left Estimate

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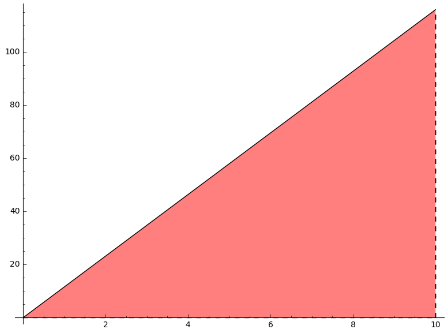
2 Equidistant Points:



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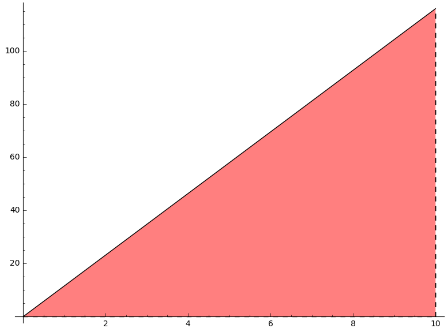
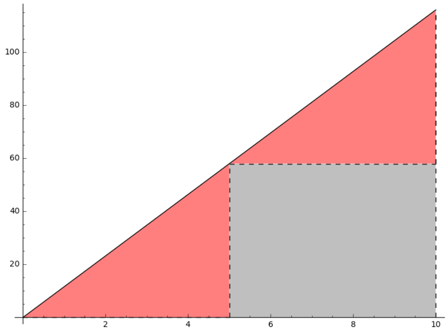


Our Estimate for the area here is **zero**. We have **underestimated** the area by $\frac{1}{2}v(t) \cdot t$.

The situation for a left endpoint estimate is symmetric:

3 Equidistant Points:



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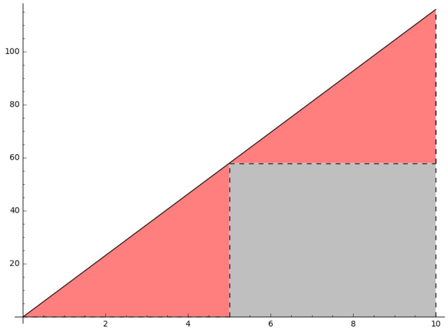
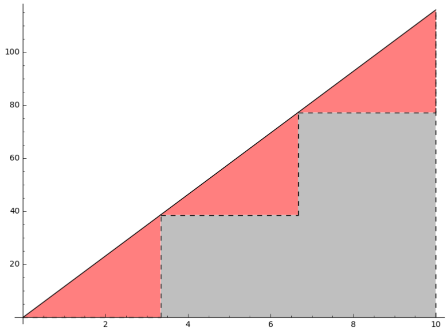


We have **underestimated** the area by $\frac{1}{2}\left(\frac{1}{2}v(t)\cdot t\right)$.

The situation for a left endpoint estimate is symmetric:

4 Equidistant Points:



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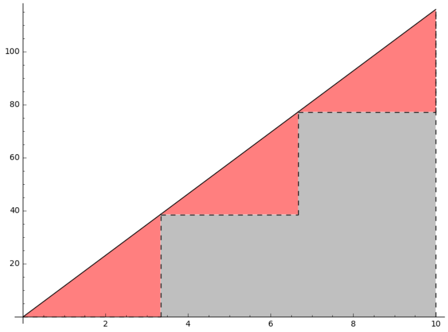

We have **underestimated** the area by $\frac{1}{3}\left(\frac{1}{2}v(t)\cdot t\right)$.

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By the same analysis as with the right estimates, using $n+1$ equidistant points

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then we expect the error will be sum of the areas of n triangles.

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Remark 2

Note that $v(t_0) = v(0) = 0$.

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Therefore, if we use $n+1$ equidistant points, we have **underestimated** the area under $v(t)$ by

$$\frac{1}{n}\left(\frac{1}{2}v(t)\cdot t\right).$$

More Is Better

- Using $n+1$ points for either a left or a right estimate, the absolute value of the error in estimating the area under the curve between 0 and $t = 10$ is given by

$$\frac{1}{n}\left(\frac{1}{2}v(t)\cdot t\right) = \frac{1}{n}\left(\frac{11.59}{2}100\right).$$

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- This tells us that as n becomes large, the error decreases. That is, the more points, the better the estimate!

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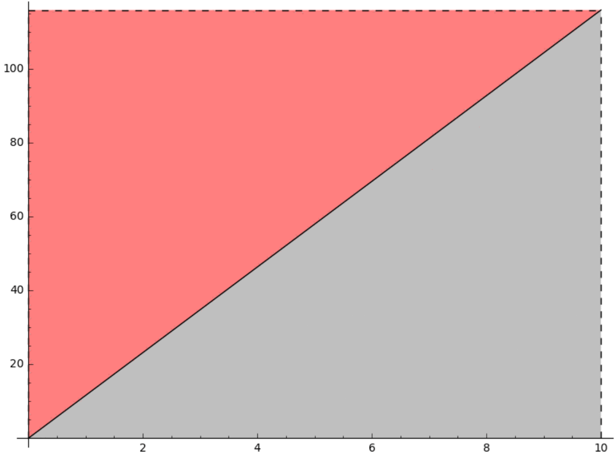
- Using $n+1$ points for either a left or a right estimate, the absolute value of the error in estimating the area under the curve between 0 and $t=10$ is given by

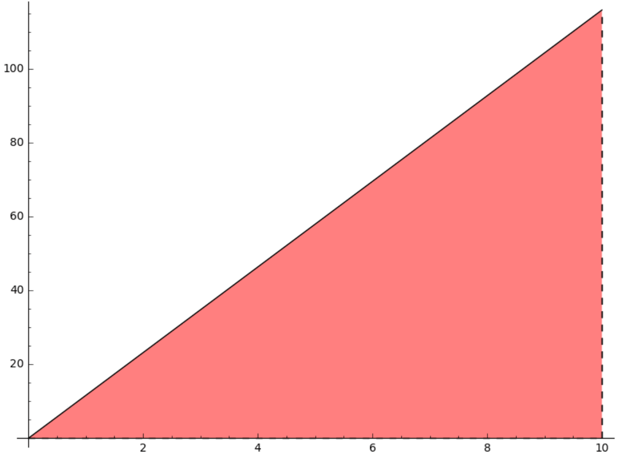
$$\frac{1}{n}\left(\frac{1}{2}v(t)\cdot t\right)=\frac{1}{n}\left(\frac{11.59}{2}100\right).$$

- This tells us that as n becomes large, the error decreases. That is, the more points, the better the estimate!
- As n grows larger, the right estimate **decreases** towards the actual area and the left estimate **increases** towards the actual area.

Right Error



Left Error



Partitions of an Interval

To generalize our methods to non-linear curves, we introduce some notation.

Definition 1

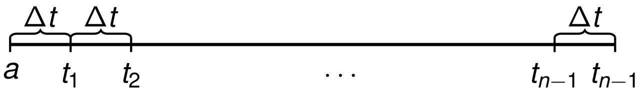
For a continuous function, f, on an interval $[a, b]$, a set of $n+1$ equidistant points,

$$t_0 = a < t_1 < t_2 < \ldots < t_{n-1} < t_n = b$$

is called a *partition* of $[a, b]$.

Partitions and Estimates

These $n+1$ points are called a partition because they partition $[a, b]$ into n smaller intervals of length Δt



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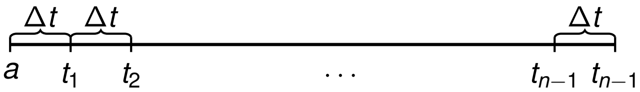

where

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These n smaller intervals form the bases of the rectangles we use to estimate the area under a curve.

Sums

Definition 2

Let f be a continuous function on the interval $[a, b]$.

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$$f(t_0)\Delta t + f(t_1)\Delta t + \cdots + f(t_{n-2})\Delta t + f(t_{n-1})\Delta t.$$

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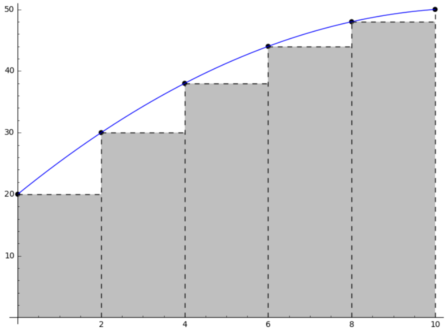
Sums (Cont.)

The Left-Hand Sum underestimates the area under the curve:



Sums (Cont.)

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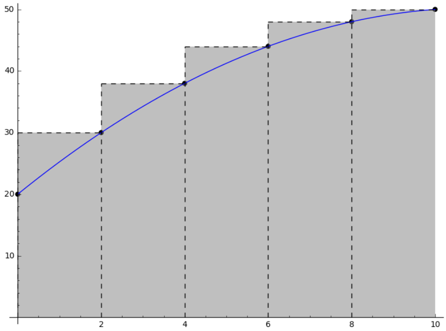

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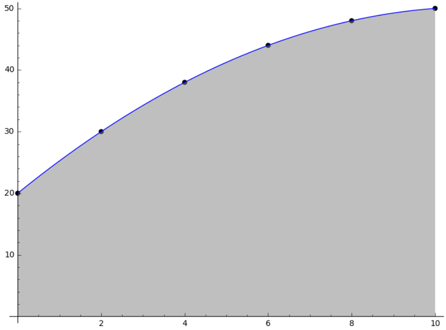
The letter i is the *index* of the summation and the letter n is the *upper bound* of the summation. The $i = 0$ underneath the sigma, Σ, indicates the sum starts at 0 and the upper bound indicates when to stop.

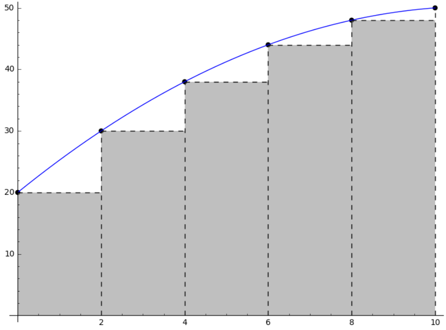
Generalizing Our Analysis

The entire point of our analysis of the linear velocity example was to improve our estimates for the non-linear curve



Generalizing our Analysis

When we use a Left-Hand Sum, we can't necessarily write down the error explicitly because the error isn't quite a triangle:



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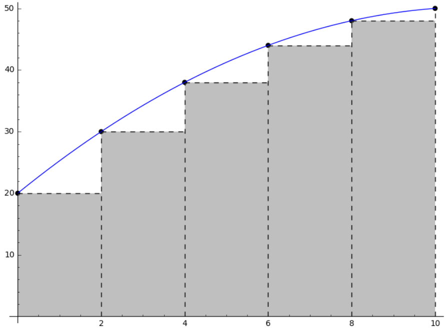

However, we can use differential calculus to get around this.

Linearization for Left-Hand Sums

Let f be a continuous function. Recall that if we take Δt sufficiently small, then we can use the Tangent Line Approximation,

$$f(t) \approx f'(a)(t - a) + f(a),$$

to ensure that f is basically a line whenever $a \leq t \leq a + \Delta t$.

Say we want to find the area beneath a continuous curve, f, on the interval $[a, b]$.

Linearization for Left-Hand Sums (cont.)

Say we want to find the area beneath a continuous curve, f, on the interval $[a, b]$.

- We can control the size of Δt by increasing the number of points in a partition

$$a = t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = b$$

since

$$\Delta t = \frac{b-a}{n}.$$

LINEARIZATION FOR LEFT-HAND SUMS (CONT.)

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- This means that if we use enough points,

$$f(t) \approx f'(t_i)(t - t_i) + f(t_i),$$

whenever $t_i \le t \le t_{i+1}$, and in particular

$$f(t_{i+1}) \approx f'(t_i)\Delta t + f(t_i).$$

Linearization for Left-Hand Sums (cont.)

Using this linearization, we get the following picture on $[t_i, t_{i+1}]$:

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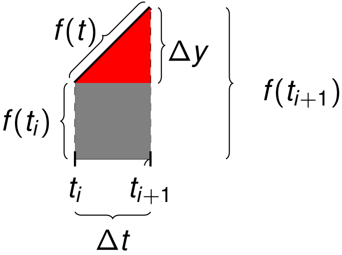

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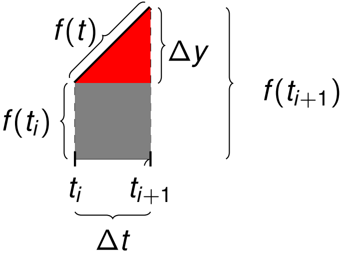

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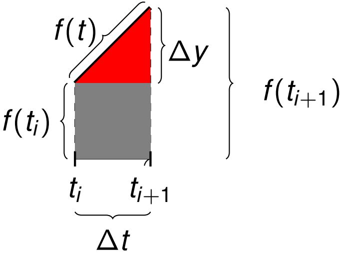

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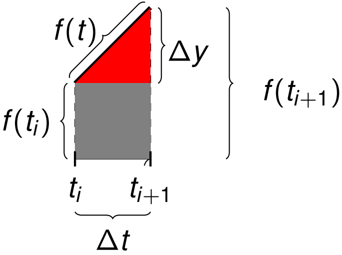

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$$\frac{1}{2}\Delta y \Delta t = \frac{1}{2}\left[f(t_{i+1}) - f(t_i)\right]\Delta t.$$

Linearization for Left-Hand Sums (cont.)

- By our work in Chapter 4, f attains a global maximum, M, and a global minimum, m, on $[a, b]$.

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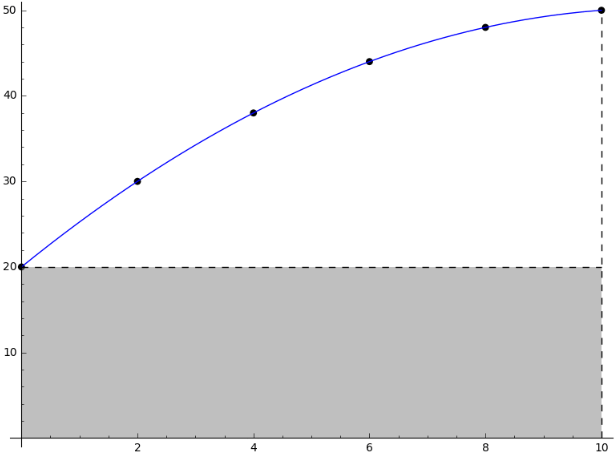
Left Sum



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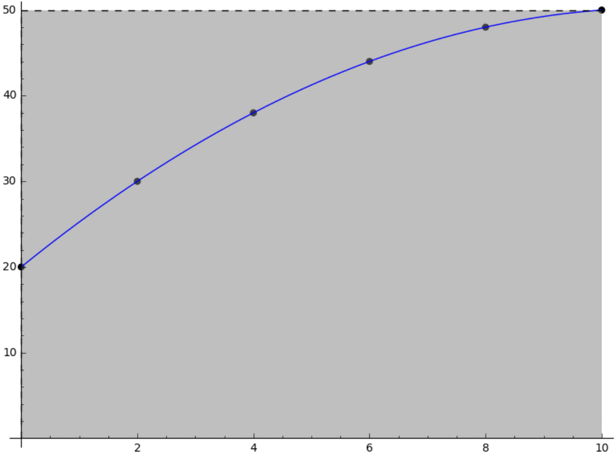
Right Sum



Our Distance Traveled Example

Recall that we started this excursion with the following question:

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Given the table of velocities and times

time (sec)	0	2	4	6	8	10
speed (ft/sec)	20	30	38	44	48	50

can we determine how far the car traveled?

Our Distance Traveled Example (Cont.)

It is possible to fit the data to the quadratic

$$v(t) = \frac{-1}{4}t^2 + \frac{11}{2}t + 20.$$

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That is,

t	0	2	4	6	8	10
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This is the curve under which we've been attempting to estimate the area. Later, we'll be able to explicitly compute that the area under this curve–which represents the distance traveled over those ten seconds–is

$$\frac{1175}{3} = 391.\overline{6} \text{ feet}$$

Our Distance Traveled Example (cont.)

With 5 equidistant points

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- Our Left-Hand Sum estimated 360 feet,

Our Distance Traveled Example (cont.)

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- Our Left-Hand Sum estimated 360 feet,
- Our Right-Hand Sum estimated 420 feet,

Our Distance Traveled Example (Cont.)

With 5 equidistant points

- Our Left-Hand Sum estimated 360 feet,
- Our Right-Hand Sum estimated 420 feet,
- Our average estimated 390 feet, which was quite close.

Our Distance Traveled Example (cont.)

Here is a table of Left-Hand Sums for $n+1$ points:

n	$\sum_{i=0}^{n-1} f(t_i)\Delta t$

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100	390.1625

Our Distance Traveled Example (Cont.)

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So we can see that as n increases, the Left-Hand Sums increase towards the actual area under the curve, as expected.

Our Distance Traveled Example (Cont.)

Here is a table of Right-Hand Sums for $n+1$ points:

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