



MATH 122

FARMAN

2.1: INSTANTANEOUS  
RATE OF  
CHANGE

2.2: THE  
DERIVATIVE  
FUNCTION

# MATH 122

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Calculus for Business Administration and Social  
Sciences



# OUTLINE

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## 1 2.1: INSTANTANEOUS RATE OF CHANGE



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2 2.2: THE DERIVATIVE FUNCTION



# DEFINITION

MATH 122

FARMAN

2.1: INSTANTANEOUS RATE OF CHANGE

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## DEFINITION 1

The *instantaneous rate of change* of  $f$  at  $a$  is defined to be the limit of the average rates of change of  $f$  over successively smaller intervals around  $a$ .

This is also known as the *derivative of  $f$  at  $a$* .



# EXAMPLE

MATH 122

FARMAN

2.1: INSTANTANEOUS RATE OF CHANGE

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The quadratic

$$s(t) = -4.9t^2 + 9.8t$$

models the position of an object thrown vertically into the air with an initial velocity of 9.8 m/s.



# EXAMPLE

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The quadratic

$$s(t) = -4.9t^2 + 9.8t$$

models the position of an object thrown vertically into the air with an initial velocity of 9.8 m/s. What is the instantaneous rate of change at the vertex, where  $t = 1$ ?



# EXAMPLE (CONT.)

MATH 122

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CHANGE

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Here are some values:

$t$

$$\frac{f(t) - f(1)}{t - 1}$$



# EXAMPLE (CONT.)

MATH 122

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Here are some values:

$t$	$\frac{f(t)-f(1)}{t-1}$
0	4.9





# EXAMPLE (CONT.)

MATH 122

FARMAN

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Here are some values:

$t$	$\frac{f(t)-f(1)}{t-1}$
0	4.9
0.9	$\approx 0.49$



# EXAMPLE (CONT.)

MATH 122

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$t$	$\frac{f(t)-f(1)}{t-1}$
0	4.9
0.9	$\approx 0.49$
0.99	$\approx 0.049$



# EXAMPLE (CONT.)

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0	4.9
0.9	$\approx 0.49$
0.99	$\approx 0.049$
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# EXAMPLE (CONT.)

MATH 122

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Here are some values:

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0.999	$\approx 0.0049$
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# EXAMPLE (CONT.)

MATH 122

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MATH 122

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0.9999	$\approx 0.00049$
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So, we would guess that the instantaneous rate of change is 0 at  $t = 1$ .



# ANIMATION

MATH 122

FARMAN

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## DEFINITION 2

- If a function,  $f$ , has a derivative at every point in its domain, then we say that  $f$  is *differentiable*.



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- If a function,  $f$ , has a derivative at every point in its domain, then we say that  $f$  is *differentiable*.
- In this case, we can define a function  $f'(x)$  that outputs the instantaneous rate of change of  $f$  at  $x$ .



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- We call  $f'(x)$  the *derivative function*.



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MATH 122

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## DEFINITION 3

- We can regard  $f'(x_0)$  as a velocity by viewing it as the slope of a line passing through  $(x_0, f(x_0))$ .



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- We can regard  $f'(x_0)$  as a velocity by viewing it as the slope of a line passing through  $(x_0, f(x_0))$ .
- We call the line

$$y - f(x_0) = f'(x_0)(x - x_0)$$

the *line tangent to  $f$  at  $(x_0, f(x_0))$* .



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MATH 122

FARMAN

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- Since we defined  $f'(x_0)$  by a limit,

$$f'(x_0) \approx \frac{f(x) - f(x_0)}{x - x_0}$$

for  $x$  close to  $x_0$ .



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- Writing  $\Delta x = x - x_0$  we can get a good linear approximation of  $f$  close to  $x_0$ :

$$f(x) \approx f'(x)\Delta x + f(x_0)$$

called the *Tangent Line Approximation*.



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- This means  $f$  locally looks like a line!





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# NON-DIFFERENTIABLE FUNCTION

MATH 122

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Consider the absolute value function

$$|x| = \begin{cases} x & \text{if } 0 \leq x, \\ -x & \text{else} \end{cases}$$

at the point  $(0, 0)$ .



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- For all  $x < 0$ ,

$$\frac{|x| - 0}{x - 0} = \frac{-x}{x} = -1.$$



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- For all  $0 < x$ ,

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- So the derivative at  $(0, 0)$  is **not** defined: it's -1 if we approach from left to right, and 1 if right to left.



MATH 122

FARMAN

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What does the derivative tells us about the original function? On the interval  $(a, b)$ , if for all  $a \leq x \leq b$



MATH 122

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- $0 \leq f'(x)$ , then  $f$  is increasing on  $(a, b)$ ,
- $f'(x) = 0$ , then  $f$  is constant on  $(a, b)$ .