Math 122

Blake Farman [1]

[1]University of South Carolina, Columbia, SC USA

Calculus for Business Administration and Social Sciences

Outline

Outline

DEFINITION 1

Let p be a point in the domain of f and let (a, b) be an interval containing p.

DEFINITION 1

Let p be a point in the domain of f and let (a, b) be an interval containing p.

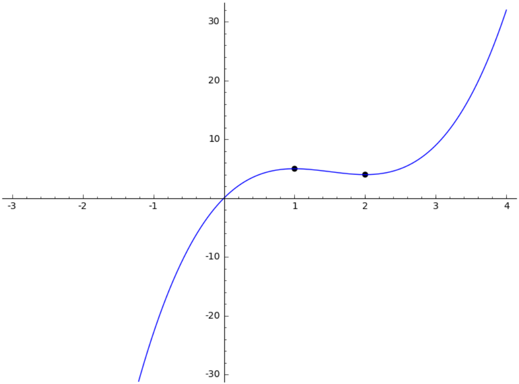
- If $f(p) \leq f(x)$ for every x satisfying $a < x < b$, then p is a *local minimum*.

Definition 1

Let p be a point in the domain of f and let (a, b) be an interval containing p.

- If $f(p) \leq f(x)$ for every x satisfying $a < x < b$, then p is a *local minimum*.
- If $f(x) \leq f(p)$ for every x satisfying $a < x < b$, then p is a *local maximum*.

Example



The point on the left is a local minimum, and the point on the right is a local maximum.

DETECTING LOCAL MAXIMA

Let f be continuous with continuous derivative, and let p be a local maximum.

Detecting Local Maxima

Let f be continuous with continuous derivative, and let p be a local maximum.

- To the left of p, f is increasing, and to the right of p, f is decreasing.

Detecting Local Maxima

Let f be continuous with continuous derivative, and let p be a local maximum.

- To the left of p, f is increasing, and to the right of p, f is decreasing.
- Equivalently:

$$0 < f'(x) \text{ for } x < p \quad \text{and} \quad f'(x) < 0 \text{ for } p < x.$$

Detecting Local Maxima

Let f be continuous with continuous derivative, and let p be a local maximum.

- To the left of p, f is increasing, and to the right of p, f is decreasing.
- Equivalently:
$$0 < f'(x) \text{ for } x < p \quad \text{and} \quad f'(x) < 0 \text{ for } p < x.$$
- Continuity of f' guarantees that $f'(p) = 0$.

Detecting Local Minima

Let f be continuous with continuous derivative, and let p be a local minimum.

Detecting Local Minima

Let f be continuous with continuous derivative, and let p be a local minimum.

- To the left of p, f is decreasing, and to the right of p, f is increasing.

Detecting Local Minima

Let f be continuous with continuous derivative, and let p be a local minimum.

- To the left of p, f is decreasing, and to the right of p, f is increasing.
- Equivalently:

$$f'(x) < 0 \text{ for } x < p \quad \text{and} \quad 0 < f'(x) \text{ for } p < x.$$

Detecting Local Minima

Let f be continuous with continuous derivative, and let p be a local minimum.

- To the left of p, f is decreasing, and to the right of p, f is increasing.
- Equivalently:

$$f'(x) < 0 \text{ for } x < p \quad \text{and} \quad 0 < f'(x) \text{ for } p < x.$$

- Continuity of f' guarantees that $f'(p) = 0$.

Example

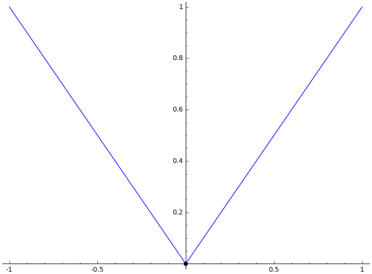
It's not always true that local extrema occur at zeroes of the first derivative.

Example

It's not always true that local extrema occur at zeroes of the first derivative. The absolute value function has a local minimum at $(0, 0)$:

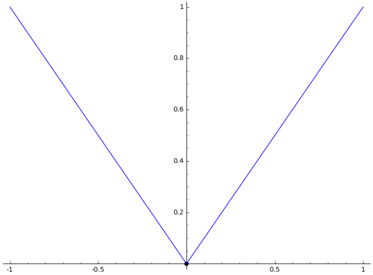
Example

It's not always true that local extrema occur at zeroes of the first derivative. The absolute value function has a local minimum at $(0, 0)$:



Example

It's not always true that local extrema occur at zeroes of the first derivative. The absolute value function has a local minimum at $(0, 0)$:



But the derivative is undefined at 0.

Critical Points

Definition 2

For a function, f, a point p in the domain of f is called a *critical point* if either

- $f'(p) = 0$, or

Critical Points

Definition 2

For a function, f, a point p in the domain of f is called a *critical point* if either

- $f'(p) = 0$, or
- $f'(p)$ is undefined.

Critical Points

Definition 2

For a function, f, a point p in the domain of f is called a *critical point* if either

- $f'(p) = 0$, or
- $f'(p)$ is undefined.

A critical value of f is the function value, $f(p)$, at a critical point, p.

Detecting Local Extrema

Theorem 1

If a continuous function, f, has a local minimum or local maximum at p, then p is a critical point of f, provided that the domain of f is not a closed interval.

Detecting Local Extrema

Theorem 1

If a continuous function, f, has a local minimum or local maximum at p, then p is a critical point of f, provided that the domain of f is not a closed interval.

Remark 1

The converse is **FALSE**. The point $x = 0$ is a critical point of $f(x) = x^2$, but neither a local minimum nor a local maximum.

First Derivative Test

Let f be a continuous function and let p be a critical point of f.

First Derivative Test

Let f be a continuous function and let p be a critical point of f.

- If $f'(x) < 0$ for $x < p$ and $0 < f'(x)$ for $p < x$, then p is a local minimum.

FIRST DERIVATIVE TEST

Let f be a continuous function and let p be a critical point of f.

- If $f'(x) < 0$ for $x < p$ and $0 < f'(x)$ for $p < x$, then p is a local minimum.
- If $0 < f'(x)$ for $x < p$ and $f'(x) < 0$ for $p < x$, then p is a local maximum.

Second Derivative Test

Let f be a continuous function and let p be a point in the domain for which $f'(p) = 0$.

Second Derivative Test

Let f be a continuous function and let p be a point in the domain for which $f'(p) = 0$.

- If $f''(p) < 0$, then p is a local maximum,

Second Derivative Test

Let f be a continuous function and let p be a point in the domain for which $f'(p) = 0$.

- If $f''(p) < 0$, then p is a local maximum,
- If $f''(p) > 0$, then p is a local minimum,

Second Derivative Test

Let f be a continuous function and let p be a point in the domain for which $f'(p) = 0$.

- If $f''(p) < 0$, then p is a local maximum,
- If $f''(p) > 0$, then p is a local minimum,
- If $f''(p) = 0$, then the test gives no information.

Example (Quadratics)

Let $f(x) = ax^2 + bx + c$.

Example (Quadratics)

Let $f(x) = ax^2 + bx + c$.

- $f'(x)$

Example (Quadratics)

Let $f(x) = ax^2 + bx + c$.

- $f'(x) = 2ax + b$,

Example (Quadratics)

Let $f(x) = ax^2 + bx + c$.

- $f'(x) = 2ax + b$,
- $f''(x)$

Example (Quadratics)

Let $f(x) = ax^2 + bx + c$.

- $f'(x) = 2ax + b$,
- $f''(x) = 2a$,

Example (Quadratics)

Let $f(x) = ax^2 + bx + c$.

- $f'(x) = 2ax + b$,
- $f''(x) = 2a$,
- There is one critical point: $\dfrac{-b}{2a}$ (the vertex),

Example (Quadratics)

Let $f(x) = ax^2 + bx + c$.

- $f'(x) = 2ax + b$,
- $f''(x) = 2a$,
- There is one critical point: $\dfrac{-b}{2a}$ (the vertex),
- The vertex is a local maximum if $a < 0$,

Example (Quadratics)

Let $f(x) = ax^2 + bx + c$.

- $f'(x) = 2ax + b$,
- $f''(x) = 2a$,
- There is one critical point: $\dfrac{-b}{2a}$ (the vertex),
- The vertex is a local maximum if $a < 0$,
- The vertex is a local minimum if $0 < a$.

Example

Find the local extrema of the function

$$f(x) = 2x^3 - 9x^2 + 12x.$$

Example

Find the local extrema of the function

$$f(x) = 2x^3 - 9x^2 + 12x.$$

- $f'(x) =$

EXAMPLE

Find the local extrema of the function

$$f(x) = 2x^3 - 9x^2 + 12x.$$

- $f'(x) = 6x^2 - 18x + 12$

Example

Find the local extrema of the function

$$f(x) = 2x^3 - 9x^2 + 12x.$$

- $f'(x) = 6x^2 - 18x + 12 = 6(x^2 - 3x + 2)$

Example

Find the local extrema of the function

$$f(x) = 2x^3 - 9x^2 + 12x.$$

- $f'(x) = 6x^2 - 18x + 12 = 6(x^2 - 3x + 2) = 6(x-1)(x-2)$

Example

Find the local extrema of the function

$$f(x) = 2x^3 - 9x^2 + 12x.$$

- $f'(x) = 6x^2 - 18x + 12 = 6(x^2 - 3x + 2) = 6(x-1)(x-2)$
- The critical points are $x = 1$ and $x = 2$.

Example

Find the local extrema of the function

$$f(x) = 2x^3 - 9x^2 + 12x.$$

- $f'(x) = 6x^2 - 18x + 12 = 6(x^2 - 3x + 2) = 6(x-1)(x-2)$
- The critical points are $x = 1$ and $x = 2$.
- $f''(x) =$

Example

Find the local extrema of the function

$$f(x) = 2x^3 - 9x^2 + 12x.$$

- $f'(x) = 6x^2 - 18x + 12 = 6(x^2 - 3x + 2) = 6(x-1)(x-2)$
- The critical points are $x = 1$ and $x = 2$.
- $f''(x) = 12x - 18$

Example

Find the local extrema of the function

$$f(x) = 2x^3 - 9x^2 + 12x.$$

- $f'(x) = 6x^2 - 18x + 12 = 6(x^2 - 3x + 2) = 6(x-1)(x-2)$
- The critical points are $x = 1$ and $x = 2$.
- $f''(x) = 12x - 18 = 6(2x - 3)$
- $f''(1) =$

Example

Find the local extrema of the function

$$f(x) = 2x^3 - 9x^2 + 12x.$$

- $f'(x) = 6x^2 - 18x + 12 = 6(x^2 - 3x + 2) = 6(x - 1)(x - 2)$
- The critical points are $x = 1$ and $x = 2$.
- $f''(x) = 12x - 18 = 6(2x - 3)$
- $f''(1) = 6(2(1) - 3)$

Example

Find the local extrema of the function

$$f(x) = 2x^3 - 9x^2 + 12x.$$

- $f'(x) = 6x^2 - 18x + 12 = 6(x^2 - 3x + 2) = 6(x-1)(x-2)$
- The critical points are $x = 1$ and $x = 2$.
- $f''(x) = 12x - 18 = 6(2x - 3)$
- $f''(1) = 6(2(1) - 3) < 0$

Example

Find the local extrema of the function

$$f(x) = 2x^3 - 9x^2 + 12x.$$

- $f'(x) = 6x^2 - 18x + 12 = 6(x^2 - 3x + 2) = 6(x-1)(x-2)$
- The critical points are $x = 1$ and $x = 2$.
- $f''(x) = 12x - 18 = 6(2x - 3)$
- $f''(1) = 6(2(1) - 3) < 0$
- $f''(2) =$

Example

Find the local extrema of the function

$$f(x) = 2x^3 - 9x^2 + 12x.$$

- $f'(x) = 6x^2 - 18x + 12 = 6(x^2 - 3x + 2) = 6(x - 1)(x - 2)$
- The critical points are $x = 1$ and $x = 2$.
- $f''(x) = 12x - 18 = 6(2x - 3)$
- $f''(1) = 6(2(1) - 3) < 0$
- $f''(2) = 6(2(2) - 3)$

Example

Find the local extrema of the function

$$f(x) = 2x^3 - 9x^2 + 12x.$$

- $f'(x) = 6x^2 - 18x + 12 = 6(x^2 - 3x + 2) = 6(x-1)(x-2)$
- The critical points are $x = 1$ and $x = 2$.
- $f''(x) = 12x - 18 = 6(2x - 3)$
- $f''(1) = 6(2(1) - 3) < 0$
- $f''(2) = 6(2(2) - 3) > 0$

Example

Find the local extrema of the function

$$f(x) = 2x^3 - 9x^2 + 12x.$$

- $f'(x) = 6x^2 - 18x + 12 = 6(x^2 - 3x + 2) = 6(x-1)(x-2)$
- The critical points are $x = 1$ and $x = 2$.
- $f''(x) = 12x - 18 = 6(2x - 3)$
- $f''(1) = 6(2(1) - 3) < 0$
- $f''(2) = 6(2(2) - 3) > 0$
- $(1, 5)$ is a local maximum and $(2, 4)$ is a local minimum.

Definition

Definition 3

A point at which the graph of a function changes concavity is called an *inflection point*.

Definition

Definition 3

A point at which the graph of a function changes concavity is called an *inflection point*.

- If f' is differentiable on an interval containing p and $f''(p) = 0$ or f'' is undefined, then p is a possible inflection point.

Definition

Definition 3

A point at which the graph of a function changes concavity is called an *inflection point*.

- If f' is differentiable on an interval containing p and $f''(p) = 0$ or f'' is undefined, then p is a possible inflection point.
- If the signs of $f''(x_1)$ and $f''(x_2)$ are different for two points $x_1 < p$ and $p < x_2$, then p is an inflection point.

Example

Find the inflection points of

$$f(x) = x^3 - 9x^2 - 48x + 52.$$

Example

Find the inflection points of

$$f(x) = x^3 - 9x^2 - 48x + 52.$$

$$f'(x) = 3x^2 - 18x - 48$$

Example

Find the inflection points of

$$f(x) = x^3 - 9x^2 - 48x + 52.$$

$$\begin{aligned} f'(x) &= 3x^2 - 18x - 48 \\ f''(x) &= 6x - 18 = 6(x - 3) \end{aligned}$$

Example

Find the inflection points of

$$f(x) = x^3 - 9x^2 - 48x + 52.$$

$$\begin{aligned} f'(x) &= 3x^2 - 18x - 48 \\ f''(x) &= 6x - 18 = 6(x-3) \\ \Rightarrow f''(3) &= 0 \end{aligned}$$

Example

Find the inflection points of

$$f(x) = x^3 - 9x^2 - 48x + 52.$$

$$\begin{aligned} f'(x) &= 3x^2 - 18x - 48 \\ f''(x) &= 6x - 18 = 6(x-3) \\ \Rightarrow f''(3) &= 0 \\ f''(0) &= 6(0-3) < 0 \end{aligned}$$

Example

Find the inflection points of

$$f(x) = x^3 - 9x^2 - 48x + 52.$$

$$\begin{aligned} f'(x) &= 3x^2 - 18x - 48 \\ f''(x) &= 6x - 18 = 6(x-3) \\ \Rightarrow f''(3) &= 0 \\ f''(0) &= 6(0-3) < 0 \\ f''(4) &= 6(4-1) > 0 \end{aligned}$$

Example

Find the inflection points of

$$f(x) = x^3 - 9x^2 - 48x + 52.$$

$$\begin{aligned} f'(x) &= 3x^2 - 18x - 48 \\ f''(x) &= 6x - 18 = 6(x-3) \\ \Rightarrow f''(3) &= 0 \\ f''(0) &= 6(0-3) < 0 \\ f''(4) &= 6(4-1) > 0 \end{aligned}$$

Therefore $x = 3$ is an inflection point of f.

Example

The point $x = 0$ is a root of the second derivative of $f(x) = x^4$, but it is **not** an inflection point because

$$f''(x) = 12x^2$$

never changes sign.